Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order

Luc Blanchet, Guillaume Faye and Bénédicte Ponsot

Département d'Astrophysique Relativiste et de Cosmologie,

Centre National de la Recherche Scientifique (UPR 176),

Observatoire de Paris, 92195 Meudon Cedex, France

(29 avril 1998)

Abstract

We derive the gravitational field and equations of motion of compact binary systems up to the 5/2 post-Newtonian approximation of general relativity (where radiation-reaction effects first appear). The approximate post-Newtonian gravitational field might be used in the problem of initial conditions for the numerical evolution of binary black-hole space-times. On the other hand we recover the Damour-Deruelle 2.5PN equations of motion of compact binary systems. Our method is based on an expression of the post-Newtonian metric valid for general (continuous) fluids. We substitute into the fluid metric the standard stress-energy tensor appropriate for a system of two point-like particles. We remove systematically the infinite self-field of each particle by means of the Hadamard partie finie regularization.

Pacs Numbers : 04.25.-g, 04.25.Nx

I. INTRODUCTION

The two purposes of the present paper are:

(1) To obtain the gravitational field generated by a system of two point-like particles up to the so-called 5/2 post-Newtonian (2.5PN) order included, i.e. the order $(v/c)^5$ where v denotes a typical value of the orbital velocity of the system. The 2.5PN field may be useful for setting up initial conditions for the numerical study of the coalescence of two compact (neutron stars or black holes) objects [1,2].

(2) To derive from the gravitational field the Damour-Deruelle [3–7] equations of motion of (compact) binary systems at the same 2.5PN order. Because the 2.5PN term in the equations of motion represents the dominant contribution of the radiation reaction force, the Damour-Deruelle equations play a crucial role in theoretically accounting for the decreasing of the orbital period of the binary pulsar PSR 1913+16 [9–12].

In addition, the present paper is motivated by the current development of the future gravitational-wave observatories LIGO and VIRGO. Specifically the aim is to derive with sufficient post-Newtonian precision the dynamics of inspiralling compact binaries (which are among the most interesting sources to be detected by LIGO and VIRGO). Numerous authors [13–18] have shown that the orbital phase of inspiralling compact binaries should be computed for applications in LIGO/VIRGO up to (at least) the 3PN relative order. Resolving this problem requires *in particular* the binary's equations of motion at 3PN order, since they permit to derive the binary's 3PN energy entering the left-hand-side of the energy balance equation on which rests the derivation of the phase. They are also needed for the computation of the 3PN gravitational flux entering the right-hand-side of the balance equation. Thus the 2.5PN equations of motion derived in [3–7] and in the present paper are not quite sufficient for the problem of inspiralling compact binaries, but the method we propose should permit to tackle in future work the problem of generalization to the next 3PN order (see [8] for an attempt at solving this problem).

The dynamics of a binary system of point-like particles modelling compact objects was

investigated successfully up to 2.5PN order by Damour, Deruelle and collaborators [3–7], using basically a post-Minkowskian approximation scheme (i.e. $G \rightarrow 0$).

In a first paper by Bel *et al* [3] (see also [19]), the gravitational field and the equations of motion are obtained in algebraically closed-form to the second post-Minkowskian order (G^2): the field equations in harmonic coordinates are solved at first order by integrating the matter stress-energy tensor suitable to point-like sources (i.e. involving delta functions), and then the second-order gravitational field is constructed by iteration. The divergencies which arise due to the assumption of point-like particles are cured by means of a regularization process based on the Hadamard partie finie [20] (see [21] for an entry to mathematical literature). The equations of motion are obtained equivalently from the harmonicity condition to be satisfied by the metric, from the conservation of the (regularized) stress-energy tensor, or from the regularized geodesic equations.

In sequential papers [4,5] the post-Minkowskian equations of motion are developed up to the order G^2/c^5, i.e. neglecting any term of the order $1/c^6$ when $c \rightarrow \infty$ *and* any term of the order G^3 when $G \rightarrow 0$. However, it is well known [22,23] that in order to obtain the complete equations of motion to the dominant 2.5PN order of radiation reaction, the latter precision is not sufficient because of the occurence of terms coming from the third post-Minkowskian metric (G^3) which contribute to both 2PN ($1/c^4$) and 2.5PN ($1/c^5$) approximations. These terms of orders G^3/c^4 and G^3/c^5 have been added by Damour [6,7], thereby completing the 2.5PN binary's equations of motion. Let us refer to the above derivation of the dynamics of a binary system as the "post-Minkowskian" approach.

When obtaining the cubic terms within the post-Minkowskian approach [6,7] the two objects are not described by standard delta functions but rather by a Riesz kernel [24] depending on some complex parameter A. For non-zero A this kernel has an infinite spatial extension, but reduces to the Dirac distribution when $A \rightarrow 0$. The metric is defined by complex analytic continuation from the A-dependent post-Minkowskian iteration. The physical equations of motion corresponding to point-like particles are obtained from the A-dependent metric by taking the limit $A \rightarrow 0$ at the end of the computation. It has been proved [6,7]

that the limit exists up to the 2.5PN approximation (no poles $\sim 1/A$ develop to this order [25]).

The motion of two particles to the 2PN order (neglecting the radiation-reaction 2.5PN terms) is conservative, i.e. there exist ten integrals of motion corresponding to the Newtonian notions of energy, linear momentum, angular momentum and center of mass position. It has been shown [26,7] that the constants of motion can be recovered by variation of a generalized Lagrangian depending on the positions, velocities and accelerations of the bodies (recall that generically, i.e. in most coordinate systems, a 2PN Lagrangian depends upon accelerations [27]). Adding up the radiation-reaction terms, one finds that the previously obtained binary's 2PN energy decreases with time, and that there is quantitative agreement with the standard quadrupole formula [28,7] and with the observations of the binary pulsar [9–12].

Moreover, there have been two other lines of work which led to the complete 2.5PN dynamics of binary systems. One of these alternative approaches is based on the canonical Hamiltonian formulation of general relativity and the model of point-like sources. Such "Hamiltonian" approach was developed at the 2PN level in early works [29,30] (see also [31,32]) but completely understood only later [27] (see [33] for a review). Schäfer [34,35] completed the Hamiltonian approach to include the 2.5PN radiation-reaction terms (more recently the 3.5PN radiation-reaction terms have also been worked out [36]). The other method gives up the model of point-like sources and assumes from the start that the two bodies are extended, spherically symmetric and made of perfect fluid. Within such an "extended-body" approach the 2.5PN equations of motion were found [37,38] to be the same as obtained within the other treatments dealing with point-like particles (in particular the equations depend only on the two masses of the bodies, but not on their internal structure nor compactness).

In the present paper, we add what constitutes essentially a fourth approach to the derivation of the 2.5PN motion of binary systems, which can be qualified as "post-Newtonian", in contrast with the post-Minkowskian, Hamiltonian and extended-body approaches. With respect to the post-Minkowskian approach [3–7] we have essentially two differences:

(1) Instead of implementing a post-Minkowskian algorithm to the third order and performing afterwards the post-Newtonian re-expansion, we start directly from a post-Newtonian metric developed to 2.5PN order and which is valid for any continuous matter stress-energy distribution ("fluid"). Note however that our post-Newtonian metric is defined in terms of *retarded* (Minkowskian) potentials, and most importantly matches a far-zone metric satisfying the correct boundary conditions at infinity, in particular the no-incoming radiation condition [39].

(2) Instead of assuming a fictituous stress-energy tensor defined by means of analytic continuation using the Riesz kernel and letting the analytic-continuation factor going to zero at the end of the computation [6,7], we substitute directly into the "fluid" metric the stress-energy tensor of point-like particles. This entails divergencies which are cured systematically by means of the Hadamard regularization [20,21] (in this respect we follow [3]).

By implementing the post-Newtonian approach, we recover the Damour-Deruelle equations of motion [4–7]. To the investigated order we find that the post-Newtonian approach is well-defined and rather systematic.

About the gravitational field (metric) generated by the two particles, we obtain an algebraically closed-form valid everywhere in space-time up to the 2.5PN order [40]. Indeed, the most difficult term present in the metric at this order is a cubically non-linear term which can be explicitly evaluated [41,42]. This yields the other motivation of the present paper, namely to provide the metric coefficients at 2.5PN order (in harmonic coordinates) in the form of some explicit, fully reduced functionals of the positions and (coordinate) velocities of the two masses. Let us point out that, very likely, the possibility of writing such a closed-form expression of the metric breaks down at the next 3PN order, where there remain some Poisson-type integrals which probably cannot be expressed in terms of simple functions.

The plan of the paper is as follows. We start in Section II with the expression (derived in Appendix A) of the 2.5PN metric valid for general fluid systems. In Section III we explain our method for applying the fluid metric to the case of point-like particles. The metric

potentials involve three types of terms which are evaluated respectively in Sections IV, V and VI (the most difficult, cubic, term being obtained in Section VI). The results for the potentials are relegated to Appendix B. In Section VII we present our expression for the binary's gravitational field. In Section VIII we finally obtain the (Damour-Deruelle) binary's equations of motion.

II. THE 2.5PN METRIC FOR GENERAL FLUID SYSTEMS

At the basis of our investigation is the expression of the metric generated by an arbitrary matter distribution described by the stress-energy tensor $T^{\mu\nu}$ [43]. We assume that $T^{\mu\nu}$ has a spatially compact support, and physically corresponds to a slowly-moving, weakly-stressed and self-gravitating system, in the sense respectively that $|T^{0i}/T^{00}| \sim \varepsilon$, $|T^{ij}/T^{00}| \sim \varepsilon^2$ and $U/c^2 \sim \varepsilon^2$, where U denotes the Newtonian potential and ε represents a small post-Newtonian parameter going to zero when the speed of light tends to infinity ($\varepsilon \sim 1/c$). Throughout this paper we denote a post-Newtonian term of order ε^n by means of the short-hand $O(n)$.

Following [44] it is convenient to define a mass density σ which agrees in the case of stationary systems with the Tolman mass density to 1PN order. As it turns out, introducing such a mass density (and in addition the associated retarded potential) permits to formulate the 2.5PN metric in a rather simple fashion. Defining also some current and stress densities we pose

$$\sigma \equiv \frac{T^{00}+T^{ii}}{c^2} \ , \tag{2.1a}$$

$$\sigma_i \equiv \frac{T^{0i}}{c} \ , \tag{2.1b}$$

$$\sigma_{ij} \equiv T^{ij} \ . \tag{2.1c}$$

The covariant conservation of the matter stress-energy tensor ($\nabla_\nu T^{\mu\nu} = 0$) entails the equations of motion and continuity, which read (with relative 1PN precision in the equation of continuity but only Newtonian precision in the equation of motion)

$$\partial_t\sigma + \partial_i\sigma_i = \frac{1}{c^2}\left(\partial_t\sigma_{ii} - \sigma\partial_t U\right) + O(4) \ , \tag{2.2a}$$

$$\partial_t\sigma_i + \partial_j\sigma_{ij} = \sigma\partial_i U + O(2) \ , \tag{2.2b}$$

where U is given by the standard Poisson integral: $U = \Delta^{-1}\{-4\pi G\sigma\}$.

Actually, it is advantageous to use rather than the instantaneous potential U, a corresponding *retarded* potential V given by the retarded integral of the same source σ:

$$V(\mathbf{x},t) = \Box_R^{-1}\left\{-4\pi G\sigma\right\} \equiv G\int \frac{d^3\mathbf{z}}{|\mathbf{x}-\mathbf{z}|}\sigma(\mathbf{z}, t - |\mathbf{x}-\mathbf{z}|/c) \ . \tag{2.3}$$

To Newtonian order we have $V = U + O(2)$ [45]. Similarly let us introduce the following other retarded potentials [46]:

$$V_i = \Box_R^{-1}\left\{-4\pi G\sigma_i\right\} \ , \tag{2.4a}$$

$$\hat{W}_{ij} = \Box_R^{-1}\left\{-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V\right\} \ , \tag{2.4b}$$

$$\hat{R}_i = \Box_R^{-1}\left\{-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right\} \ , \tag{2.4c}$$

$$\begin{aligned}\hat{X} = \Box_R^{-1}\Big\{&-4\pi G V\sigma_{ii} + 2V_i\partial_t\partial_i V + V\partial_t^2 V \\ &+\frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j\partial_j V_i + \hat{W}_{ij}\partial_{ij}^2 V\Big\} \ .\end{aligned} \tag{2.4d}$$

In addition, we shall often consider the trace of the potential $\hat{W}_{ij}$, i.e.

$$\hat{W}_{ii} = \Box_R^{-1}\left\{8\pi G\sigma_{ii} - \partial_i V\partial_i V\right\} \ . \tag{2.4e}$$

We are now able to express the usual covariant metric $g_{\mu\nu}$ in terms of these retarded potentials to order 2.5PN, by which we mean neglecting all the terms of order $O(8)$ in g_{00}, $O(7)$ in g_{0i} and $O(6)$ in g_{ij}. We impose the harmonic or De Donder coordinate conditions, i.e. $\partial_\nu[\sqrt{-g}g^{\mu\nu}] = 0$, where g and $g^{\mu\nu}$ are the determinant and the inverse of the matrix $g_{\mu\nu}$. Actually, since we are working with an approximate post-Newtonian metric, the harmonic conditions need only to be satisfied approximately. To the 2.5PN order we have $\partial_\nu[\sqrt{-g}g^{0\nu}] = O(7)$ and $\partial_\nu[\sqrt{-g}g^{i\nu}] = O(6)$.

With these definitions the 2.5PN metric in harmonic coordinates takes the form:

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left[\hat{X} + V_i V_i + \frac{V^3}{6}\right] + O(8) \ , \tag{2.5a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + O(7) \ , \tag{2.5b}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right) + \frac{4}{c^4}\hat{W}_{ij} + O(6) \ . \tag{2.5c}$$

For the sake of completeness, this rather simple result is proved in Appendix A. The simplicity in the formulation is due to our introduction of the mass density σ as well as the use of retarded potentials [44,39].

In the form (2.5) the metric contains only "even" terms explicitly (using the standard post-Newtonian terminology), that are terms with even powers of $1/c$ in g_{00} and g_{ij}, and odd powers in g_{0i}. Indeed the "odd" terms, which are responsible for radiation reaction forces, are all hidden into the definitions of the retarded potentials (2.3)-(2.4), and can be made explicit by expanding the retarded arguments with Taylor's formula. It is important in this respect to recall from [39] that (2.5) comes from the post-Newtonian expansion (valid only in the near zone) of some radiative metric defined globally in space-time and satisfying the no-incoming radiation condition at past null infinity. Hence the "odd" terms in the post-Newtonian metric (2.5) correspond physically to the radiation reaction forces acting on an isolated system (with no source located at infinity).

At 2.5PN order the harmonic-coordinate conditions are equivalent to the following differential identities:

$$\partial_t\left\{V + \frac{1}{c^2}\left[\frac{1}{2}\hat{W}_{ii} + 2V^2\right]\right\} + \partial_i\left\{V_i + \frac{2}{c^2}\left[\hat{R}_i + VV_i\right]\right\} = O(4) \ , \tag{2.6a}$$

$$\partial_t V_i + \partial_j\left\{\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk}\right\} = O(2) \ . \tag{2.6b}$$

These relations are in turn equivalent to the 1PN continuity equation and Newtonian equation of motion given by (2.2).

The potentials V and V_i are generated by the compact-supported source densities σ and σ_i. Similarly $\hat{W}_{ij}$ and $\hat{R}_i$ involve a part generated by a compact-supported source, but also a part whose source is a sum of quadratic products of potentials V or V_i and their space-time derivatives. We shall refer to the former part of $\hat{W}_{ij}$ and $\hat{R}_i$ as the compact ("C") part, and

to the latter as the "$\partial V\partial V$" or, sometimes, "quadratic" part. As for $\hat{X}$, it consists of C and $\partial V\partial V$ parts like for $\hat{W}_{ij}$ and $\hat{R}_i$, but it also contains a term of different structure, generated by the product of $\hat{W}_{ij}$ and $\partial^2_{ij}V$ [last term in (2.4d)]. This term itself can be split into two contributions arising respectively from the C and $\partial V\partial V$ parts of $\hat{W}_{ij}$. Since the C part of $\hat{W}_{ij}$ is a compact-supported potential similar to V and V_i, the corresponding term in $\hat{X}$ has actually the same structure as a $\partial V\partial V$ potential. On the other hand, the $\partial V\partial V$ part of $\hat{W}_{ij}$ generates an intrinsically more complicated term in $\hat{X}$ we shall refer to as the non-compact ("NC") or "cubic" part of the potential $\hat{X}$. Precisely, our definitions are

$$\hat{W}_{ij} = \hat{W}^{(\mathrm{C})}_{ij} + \hat{W}^{(\partial V\partial V)}_{ij} \;, \tag{2.7a}$$

$$\hat{R}_i = \hat{R}^{(\mathrm{C})}_i + \hat{R}^{(\partial V\partial V)}_i \;, \tag{2.7b}$$

$$\hat{X} = \hat{X}^{(\mathrm{C})} + \hat{X}^{(\partial V\partial V)} + \hat{X}^{(\mathrm{NC})} \;. \tag{2.7c}$$

The compact parts are linear and quadratic functionals of the matter variables (2.1). They read as

$$\hat{W}^{(\mathrm{C})}_{ij} = \Box^{-1}_R \left\{-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk})\right\} \;, \tag{2.8a}$$

$$\hat{R}^{(\mathrm{C})}_i = \Box^{-1}_R \left\{-4\pi G(V\sigma_i - V_i\sigma)\right\} \;, \tag{2.8b}$$

$$\hat{X}^{(\mathrm{C})} = \Box^{-1}_R \left\{-4\pi G V\sigma_{ii}\right\} \;. \tag{2.8c}$$

The $\partial V\partial V$ or "quadratic" parts involve both quadratic and cubic contributions. They are

$$\hat{W}^{(\partial V\partial V)}_{ij} = \Box^{-1}_R \left\{-\partial_i V\partial_j V\right\} \;, \tag{2.9a}$$

$$\hat{R}^{(\partial V\partial V)}_i = \Box^{-1}_R \left\{-2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right\} \;, \tag{2.9b}$$

$$\begin{aligned}\hat{X}^{(\partial V\partial V)} = \Box^{-1}_R \Big\{&2V_i\partial_t\partial_i V + V\partial^2_t V + \frac{3}{2}(\partial_t V)^2 \\ &- 2\partial_i V_j\partial_j V_i + \hat{W}^{(\mathrm{C})}_{ij}\partial^2_{ij}V\Big\} \;.\end{aligned} \tag{2.9c}$$

Finally the only Non-Compact part is a cubic functional given by

$$\hat{X}^{(\mathrm{NC})} = \Box^{-1}_R \left\{\hat{W}^{(\partial V\partial V)}_{ij}\partial^2_{ij}V\right\} \;. \tag{2.10}$$

In practice the latter term is the most delicate to evaluate. Our terminology is slightly improper, as the so-called $\partial V \partial V$ or quadratic potentials are as well as the NC potential generated by non-compact supported sources, and involve some contributions which are actually cubic in the matter variables.

III. APPLICATION TO POINT-LIKE PARTICLES

To apply the general 2.5PN metric presented in the previous section to the case of a point-mass binary we use the matter stress-energy tensor:

$$T^{\mu\nu}(\mathbf{x},t) = \mu_1(t) v_1^\mu(t) v_1^\nu(t) \delta(\mathbf{x}-\mathbf{y}_1(t)) + 1 \leftrightarrow 2 \ . \tag{3.1}$$

In our notation the symbol $1 \leftrightarrow 2$ means the same term but with the labels 1 and 2 exchanged; δ denotes the three-dimensional Dirac distribution; the trajectories of the two masses (in harmonic coordinates) are denoted by $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$; the two coordinate velocities are $\mathbf{v}_1(t) = d\mathbf{y}_1(t)/dt$, $\mathbf{v}_2(t) = d\mathbf{y}_2(t)/dt$, and $v_1^\mu \equiv (c, \mathbf{v}_1)$, $v_2^\mu \equiv (c, \mathbf{v}_2)$; μ_1 represents an effective time-dependent mass of body 1 defined by

$$\mu_1(t) = \left(\frac{m_1}{\sqrt{g g_{\rho\sigma} \dfrac{v_1^\rho v_1^\sigma}{c^2}}} \right)_1 \ , \tag{3.2}$$

m_1 being the (constant) Schwarzschild mass, with $g_{\rho\sigma}$ the metric and g its determinant [47]. Another useful notation is

$$\tilde{\mu}_1(t) = \mu_1(t) \left[1 + \frac{v_1^2}{c^2} \right] \ , \tag{3.3}$$

where $v_1^2 = \mathbf{v}_1^2$. Both μ_1 and $\tilde{\mu}_1$ reduce to the Schwarszchild mass at Newtonian order: $\mu_1 = m_1 + O(2)$ and $\tilde{\mu}_1 = m_1 + O(2)$. Then the mass, current and stress densities (2.1) for two particles read

$$\sigma = \tilde{\mu}_1 \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightarrow 2 \ , \tag{3.4a}$$

$$\sigma_i = \mu_1 v_1^i \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightarrow 2 \ , \tag{3.4b}$$

$$\sigma_{ij} = \mu_1 v_1^i v_1^j \delta(\mathbf{x}-\mathbf{y}_1) + 1 \leftrightarrow 2 \ . \tag{3.4c}$$

The stress-energy tensor of point-masses depends on the values of the metric coefficients at the very location of the particles. However the metric coefficients there become infinite and, consequently, we must supplement the model of stress-energy tensor (3.1) by a prescription for giving a sense to the notion of the field sitting on the particle. In other words, we need a regularization procedure in order to remove the infinite self-field of point-like sources. The choice of one or another regularization procedure represents (*a priori*) an integral part of the choice of physical model for describing the particles. In the present paper we shall employ the Hadamard regularization [20,21] based on the partie finie of functions admitting a special ("tempered") type of singularity. For discussion and justification of the use of the Hadamard regularization in the context of equations of motion in general relativity see [3,5,34,42,8,48].

Let us consider the class of functions F depending on the field point $\mathbf{x}$ as well as on two source points $\mathbf{y}_1$ and $\mathbf{y}_2$, and admitting when the field point approaches one of the source points ($r_1 = |\mathbf{x} - \mathbf{y}_1| \to 0$ for instance) an expansion of the type

$$F(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2) = \sum_{-k_0 \le k \le 0} r_1^k f_k(\mathbf{n}_1;\mathbf{y}_1,\mathbf{y}_2) + O(r_1) \tag{3.5}$$

(where $k \in \mathbb{Z}$). We define the value of the function F at the source point 1 (and similarly at the source point 2) to be the so-called Hadamard partie finie, which is the average, with respect to the direction $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ of approach to point 1, of the term with zeroth power of r_1 in (3.5). Namely [49],

$$(F)_1 \equiv F(\mathbf{y}_1;\mathbf{y}_1,\mathbf{y}_2) \equiv \int \frac{d\Omega(\mathbf{n}_1)}{4\pi} f_0(\mathbf{n}_1;\mathbf{y}_1,\mathbf{y}_2) \ . \tag{3.6}$$

Furthermore, we use the Hadamard partie finie to give a sense to the spatial integral of the product of F and the Dirac delta function at point 1 (since F is singular on the support of the Dirac function). Indeed, we define

$$\int d^3\mathbf{x} F(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\delta(\mathbf{x} - \mathbf{y}_1) \equiv (F)_1 \ , \tag{3.7}$$

where $(F)_1$ is given by (3.6).

As a (trivial) example of the use of the Hadamard regularization, consider the potentials V and V_i to Newtonian order, given by

$$V = \frac{Gm_1}{r_1} + O(2) + 1 \leftrightarrow 2 \ , \tag{3.8a}$$

$$V_i = \frac{Gm_1}{r_1} v_1^i + O(2) + 1 \leftrightarrow 2 \ . \tag{3.8b}$$

They are infinite at point 1, but after applying the rule (3.6) we find

$$(V)_1 = \frac{Gm_2}{r_{12}} + O(2) \ , \tag{3.9a}$$

$$(V_i)_1 = \frac{Gm_2}{r_{12}} v_2^i + O(2) \ , \tag{3.9b}$$

where $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$ is the distance between the particles [50]. Of course $(V)_1$ agrees with the standard Newtonian result. Applying the rule (3.7) we have, for instance,

$$\frac{1}{2} \int d^3\mathbf{x} \ \sigma V = \frac{Gm_1 m_2}{r_{12}} + O(2) \ , \tag{3.10}$$

also in agreement with the Newtonian result.

We shall derive the binary's equations of motion in the so-called order-reduced form, by which we mean that in the final result all accelerations (and time-derivatives of accelerations) are replaced consistently with the approximation by the explicit functionals of the positions and velocities as given by the (lower-order) equations. So, in order to derive the 2.5PN equations of motion (and also the metric), we use the less accurate 1.5PN equations, given in harmonic coordinates by

$$\begin{aligned} \frac{dv_1^i}{dt} = & -\frac{Gm_2}{r_{12}^2} n_{12}^i \left\{ 1 + \frac{1}{c^2} \left[-5 \frac{Gm_1}{r_{12}} - 4 \frac{Gm_2}{r_{12}} + v_1^2 + 2v_2^2 - 4(v_1 v_2) - \frac{3}{2} (n_{12} v_2)^2 \right] \right\} \\ & + \frac{Gm_2}{c^2 r_{12}^2} v_{12}^i \left[4(n_{12} v_1) - 3(n_{12} v_2) \right] + O(4) \ , \end{aligned} \tag{3.11}$$

with $\mathbf{n}_{12} = |\mathbf{y}_1 - \mathbf{y}_2|/r_{12}$ and $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$; scalar products are denoted with parenthesis, e.g. $(n_{12} v_1) = \mathbf{n}_{12}.\mathbf{v}_1$. The acceleration of body 2 is obtained by exchanging the labels 1 and 2 (remembering that $\mathbf{n}_{12}$ and $\mathbf{v}_{12}$ change sign in this operation).

IV. THE COMPACT PARTS OF POTENTIALS

In this section we derive the compact-supported potentials V and V_i, and the compact-supported parts of the other potentials, $\hat{W}_{ij}^{(\mathrm{C})}$, $\hat{R}_i^{(\mathrm{C})}$ and $\hat{X}^{(\mathrm{C})}$ defined by (2.8), for a binary system described by the stress-energy tensor (3.1) and the regularization (3.6). We need V to relative 2.5PN order, V_i to 1.5PN order, and the other compact potentials to 0.5PN order only. The main task is the computation of V, to which we focus mainly our attention. By Taylor-expanding at 2.5PN order the retardation inside the integral (2.3) and using the mass density σ in the form (3.4a), we get

$$V = G\left\{\frac{\tilde{\mu}_1}{r_1} - \frac{1}{c}\partial_t(\tilde{\mu}_1) + \frac{1}{2c^2}\partial_t^2(\tilde{\mu}_1 r_1) - \frac{1}{6c^3}\partial_t^3(\tilde{\mu}_1 r_1^2) \right.$$
$$\left. + \frac{1}{24c^4}\partial_t^4(\tilde{\mu}_1 r_1^3) - \frac{1}{120c^5}\partial_t^5(\tilde{\mu}_1 r_1^4)\right\} + O(6) + 1 \leftrightarrow 2\ . \tag{4.1}$$

We recall that the effective mass $\tilde{\mu}_1$ given by (3.2)-(3.3) is a function of time only.

We start by deriving $\tilde{\mu}_1$ to 2.5PN order. Inserting the metric coefficients (2.5) into the expressions (3.2)-(3.3), we obtain

$$\tilde{\mu}_1 = m_1\left\{1 + \frac{1}{c^2}\left[-(V)_1 + \frac{3}{2}v_1^2\right]\right.$$
$$\left. + \frac{1}{c^4}\left[-2(\hat{W}_{ii})_1 + \frac{1}{2}(V^2)_1 + \frac{1}{2}(V)_1 v_1^2 - 4(V_i)_1 v_1^i + \frac{7}{8}v_1^4\right]\right\} + O(6)\ , \tag{4.2}$$

where all the potentials are to be evaluated at the location of body 1, using the rule (3.6). We proceed iteratively. The first step consists in inserting into (4.2) the potential V at body 1 to Newtonian order (or, rather, 0.5PN order), which is simply the Newtonian result (3.9a). This yields $\tilde{\mu}_1$ to 1.5PN order:

$$\tilde{\mu}_1 = m_1\left\{1 + \frac{1}{c^2}\left[-\frac{Gm_2}{r_{12}} + \frac{3}{2}v_1^2\right]\right\} + O(4)\ . \tag{4.3}$$

Since the time-derivative of $\tilde{\mu}_1$ starts at 1PN order, namely

$$\dot{\tilde{\mu}}_1 = \frac{Gm_1m_2}{c^2 r_{12}^2}\left[-2(n_{12}v_1) - (n_{12}v_2)\right] + O(4)\ , \tag{4.4}$$

we see that the first odd power of $1/c$ in V arises at 1.5PN order. Furthermore, using the constancy of the center of mass velocity, one can check that the first odd term in the gradient

of V arises at 2.5PN order (it contributes to the dominant radiation reaction force). From (4.1) and (4.3) we deduce the value of $(V)_1$ up to 1.5PN order:

$$\begin{aligned}(V)_1 = &\frac{Gm_2}{r_{12}}\left\{1+\frac{1}{c^2}\left[-\frac{3Gm_1}{2r_{12}}+2v_2^2-\frac{1}{2}(n_{12}v_2)^2\right]\right\} \\ &+\frac{4G^2m_1m_2}{3c^3r_{12}^2}(n_{12}v_{12})+O(4)\ .\end{aligned} \tag{4.5}$$

In addition to $(V)_1$, we need $(V_i)_1$ already given by (3.9b), and the value at point 1 of the trace of $\hat{W}_{ij}$ to 0.5PN order. The trace $\hat{W}_{ii}$ is much simpler than the potential itself, and from (2.4e) we derive the expression

$$\begin{aligned}\hat{W}_{ii} = &\Delta^{-1}\left\{8\pi G\left(\sigma_{ii}-\frac{1}{2}\sigma V\right)\right\}-\frac{1}{2}V^2 \\ &+\frac{2G}{c}\frac{d}{dt}\int d^3\mathbf{x}\left(\sigma_{ii}-\frac{1}{2}\sigma V\right)+O(2)\ .\end{aligned} \tag{4.6}$$

Under this form all integrals are compact-supported; at this order, we can insert $V = U+O(2)$. The odd term $O(1)$ is a mere function of time. From (4.6) we get immediately

$$(\hat{W}_{ii})_1 = \frac{Gm_2}{r_{12}}\left[\frac{Gm_1}{r_{12}}-\frac{Gm_2}{2r_{12}}-2v_2^2\right]-\frac{2G^2m_1m_2}{cr_{12}^2}(n_{12}v_{12})+O(2)\ . \tag{4.7}$$

The effective mass $\tilde{\mu}_1$ at 2.5PN order is readily obtained from the previous relations:

$$\begin{aligned}\tilde{\mu}_1 = m_1\Bigg\{&1+\frac{1}{c^2}\left[-\frac{Gm_2}{r_{12}}+\frac{3}{2}v_1^2\right] \\ &+\frac{1}{c^4}\left[\frac{Gm_2}{r_{12}}\left(\frac{1}{2}v_1^2-4(v_1v_2)+2v_2^2+\frac{1}{2}(n_{12}v_2)^2-\frac{1}{2}\frac{Gm_1}{r_{12}}+\frac{3}{2}\frac{Gm_2}{r_{12}}\right)+\frac{7}{8}v_1^4\right] \\ &+\frac{8G^2m_1m_2}{3c^5r_{12}^2}(n_{12}v_{12})\Bigg\}+O(6)\ ,\end{aligned} \tag{4.8}$$

from which we straightforwardly deduce V to 2.5PN order. The only point is to compute the numerous (up to five) time-derivatives of r_1. This gives rise to many terms depending on the acceleration and its time-derivatives, that we reduce order by order by means of the binary's 1.5PN equations of motion (since an acceleration already arises in the 1PN term of V). Once fully reduced, the result for V is still lengthy so we relegate it (together with all the relevant results for the potentials) in Appendix B.

The potential V_i to 1.5PN order, and the other compact potentials $\hat{W}_{ij}^{(\mathrm{C})}$, $\hat{R}_i^{(\mathrm{C})}$ and $\hat{X}^{(\mathrm{C})}$ to 0.5PN order, are obtained in the same way. As an example we give

$$\hat{W}^{(\mathrm{C})}_{ij} = \frac{Gm_1}{r_1}\left(v_1^i v_1^j - \delta^{ij} v_1^2\right) + \frac{G^2 m_1 m_2}{c r_{12}^2}\left[n_{12}^{(i} v_{12}^{j)} - \delta^{ij}(n_{12} v_{12})\right] + O(2) + 1 \leftrightarrow 2 \ . \tag{4.9}$$

V. THE QUADRATIC PARTS OF POTENTIALS

By the definition (2.9) the quadratic or $\partial V \partial V$ potentials have their sources made of quadratic products of (derivatives of) the compact-supported potentials V, V_i and $\hat{W}^{(\mathrm{C})}_{ij}$. All the $\partial V \partial V$ potentials are to be computed to 0.5PN order, which means in particular that we can replace in the sources V and V_i by the Newtonian-like potentials U and U_i (but we must beware of the fact that $\hat{W}^{(\mathrm{C})}_{ij}$ given by (4.9) involves a $1/c$ correction). For all the $\partial V \partial V$ potentials we proceed similarly. We work out the sources using (3.8) and (4.9) and obtain some "self" terms, proportional to m_1^2 and m_2^2, together with some "interaction" terms, proportional to $m_1 m_2$. Time-derivatives are changed to spatial derivatives thanks to $\partial_t(1/r_1) = v_1^i \partial_{1i}(1/r_1)$ and $\partial_t^2(1/r_1) = a_1^i \partial_{1i}(1/r_1) + v_1^i v_1^j \partial_{1ij}(1/r_1)$, a_1^i denoting the acceleration and ∂_{1i} the partial derivative with respect to y_1^i. In the interaction terms, we leave the spatial derivatives un-expanded, whereas, in the self terms, they are developed and "factorized" out in front of the terms. In the latter operation, we should remember that within the standard distribution theory the second spatial derivative of $1/r_1$ involves a distributional term [51,21,48]:

$$\Delta\left(\frac{1}{r_1}\right) = -4\pi\delta(\mathbf{x}-\mathbf{y}_1) \ , \tag{5.1a}$$

$$\partial^2_{ij}\left(\frac{1}{r_1}\right) = \frac{3n_1^i n_1^j - \delta^{ij}}{r_1^3} - \frac{4\pi}{3}\delta^{ij}\delta(\mathbf{x}-\mathbf{y}_1) \ . \tag{5.1b}$$

Two examples of such a treatment of sources are

$$\begin{aligned}\partial_i V \partial_j V &= \frac{G^2 m_1^2}{8}\left(\partial^2_{1ij} + \delta^{ij}\Delta_1\right)\left(\frac{1}{r_1^2}\right) \\ &\quad + G^2 m_1 m_2 \partial_{1i}\partial_{2j}\left(\frac{1}{r_1 r_2}\right) + O(2) + 1 \leftrightarrow 2 \ ,\end{aligned} \tag{5.2a}$$

$$\begin{aligned}V \partial_t^2 V &= G^2 m_1^2 \left[\frac{1}{8}\left(4a_1^i \partial_{1i} + 3v_1^{ij}\partial^2_{1ij} - v_1^2 \Delta_1\right)\left(\frac{1}{r_1^2}\right) - \frac{4\pi}{3}\frac{v_1^2}{r_1}\delta(\mathbf{x}-\mathbf{y}_1)\right] \\ &\quad + G^2 m_1 m_2 \left(a_1^i \partial_{1i} + v_1^{ij}\partial^2_{1ij}\right)\left(\frac{1}{r_1 r_2}\right) + O(2) + 1 \leftrightarrow 2 \ .\end{aligned} \tag{5.2b}$$

We apply the Poisson integral on the source terms treated in the previous manner. Consider firstly the distributional terms, such as the one in the self part of $V\partial_t^2 V$. Although this term is ill-defined, because involving the product of the Dirac distribution $\delta(\mathbf{x}-\mathbf{y}_1)$ by $1/r_1$ which is singular when $\mathbf{x}\to\mathbf{y}_1$, the Poisson integral is computed unambiguously with the help of the Hadamard regularization (3.7), yielding zero in this case:

$$\begin{aligned}\Delta^{-1}\left[-\frac{4\pi}{r_1}\delta(\mathbf{x}-\mathbf{y}_1)\right] &= \int \frac{d^3\mathbf{z}}{|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}_1|}\delta(\mathbf{z}-\mathbf{y}_1) \\ &= \left(\frac{1}{|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}_1|}\right)_{\mathbf{z}=\mathbf{y}_1} = 0 \ . \end{aligned} \tag{5.3}$$

For the computation of all non-distributional terms in the $\partial V\partial V$ potentials, we take the example of

$$\begin{aligned}\hat{W}_{ij}^{(\partial V\partial V)} &= \Box_R^{-1}\left\{-\partial_i V\partial_j V\right\} \\ &= \Delta^{-1}\left\{-\partial_i V\partial_j V\right\} + \frac{1}{4\pi c}\frac{d}{dt}\int d^3\mathbf{x}\ \left\{-\partial_i V\partial_j V\right\} + O(2) \ , \end{aligned} \tag{5.4}$$

whose “source” is given by (5.2a). The Poisson integral of the self-terms can be readily deduced from $\Delta(\ln r_1) = 1/r_1^2$; on the other hand, that of the interaction terms is obtained by solving the elementary Poisson equation

$$\Delta g = \frac{1}{r_1 r_2} \ . \tag{5.5}$$

The solution is known [52]:

$$g = \ln S \ ; \qquad S \equiv r_1 + r_2 + r_{12} \ . \tag{5.6}$$

The computation of the $1/c$-term in (5.4) involves essentially the spatial integral of $1/r_1r_2$. Since it is divergent due to the bound at infinity (i.e. when $r\equiv|\mathbf{x}|\to\infty$), we first compute the finite integral defined by integration over a ball of constant finite radius $\mathcal{R}$. By writing the integrand as $1/r_1r_2 = \Delta g$ and using the Gauss theorem, we transform the integral into a surface integral over the sphere of radius $\mathcal{R}$,

$$\int_{|\mathbf{x}|\leq\mathcal{R}} \frac{d^3\mathbf{x}}{r_1 r_2} = \int_{|\mathbf{x}|\leq\mathcal{R}} d^3\mathbf{x}\ \Delta g = \int_{r=\mathcal{R}} d\Omega\ (r^2\partial_r g) \ , \tag{5.7}$$

with $\partial_r \equiv n^i \partial_i$. Into the latter surface integral we can replace the function g by its expansion at infinity computed from (5.6): $g = \ln(2r) + [-(ny_1) - (ny_2) + r_{12}]/2r + O(1/r^2)$. Neglecting the terms which die out in the limit $\mathcal{R} \to \infty$, we get

$$-\frac{1}{4\pi}\int_{|\mathbf{x}|\leq\mathcal{R}} \frac{d^3\mathbf{x}}{r_1 r_2} = -\mathcal{R} + \frac{r_{12}}{2} + O\left(\frac{1}{\mathcal{R}}\right) . \tag{5.8}$$

As we can see, the divergent part of the integral is simply a constant, which will therefore vanish after application of the spatial derivatives $\partial_{1i}\partial_{2j}$ in front of the term. This shows that we are allowed to use in this computation the finite coefficient in the right-hand-side of (5.8), which is nothing but the partie finie in the sense of Hadamard of the initially divergent integral [53]. Setting $\mathbf{y}_1 = \mathbf{y}_2$ in (5.8), we infer that the divergent integral of $1/r_1^2$ can be replaced by zero. Then, (5.8) together with the last fact lead to

$$-\frac{1}{4\pi}\int d^3\mathbf{x}\, \partial_i V \partial_j V = \frac{G^2 m_1 m_2}{r_{12}}\left(n_{12}^{ij} - \delta^{ij}\right) + O(2) . \tag{5.9}$$

Gathering those results, we thereby obtain the looked-for potential as

$$\begin{aligned}\hat{W}_{ij}^{(\partial V \partial V)} = & -\frac{G^2 m_1^2}{8}\left(\partial_{ij}^2 \ln r_1 + \frac{\delta^{ij}}{r_1^2}\right) - G^2 m_1 m_2 \, {}_i g_j \\ & + \frac{G^2 m_1 m_2}{c r_{12}^2}\left[n_{12}^{(i} v_{12}^{j)} - \frac{1}{2}\left(3 n_{12}^{ij} - \delta^{ij}\right)(n_{12} v_{12})\right] + O(2) + 1 \leftrightarrow 2 ,\end{aligned} \tag{5.10}$$

where we have ${}_i g_j \equiv \partial_{1i}\partial_{2j} g$. The first two terms are in agreement with a result of [42]. All the $\partial V \partial V$-potentials are calculated in this way to obtain the complete expressions of potentials presented in Appendix B.

Ending this section we list some formulas which are useful in the derivation of the $\partial V \partial V$-potentials, and even elsewhere. The first-order spatial derivatives of g read

$${}_i g \equiv \partial_{1i} g = \frac{-n_1^i + n_{12}^i}{S} , \tag{5.11a}$$

$$g_i \equiv \partial_{2i} g = \frac{-n_2^i - n_{12}^i}{S} , \tag{5.11b}$$

$$\partial_i g = -{}_i g - g_i = \frac{n_1^i + n_2^i}{S} , \tag{5.11c}$$

and second-order spatial derivatives are (with $n_{12}^{ij} \equiv n_{12}^i n_{12}^j$)

$$ {}_{ij}g \equiv \partial^2_{1ij} g = -\frac{n_{12}^{ij}-\delta^{ij}}{r_{12}S} - \frac{n_1^{ij}-\delta^{ij}}{r_1 S} - \frac{(n_{12}^i - n_1^i)(n_{12}^j - n_1^j)}{S^2} \ , \tag{5.12a} $$

$$ g_{ij} \equiv \partial^2_{2ij} g = -\frac{n_{12}^{ij}-\delta^{ij}}{r_{12}S} - \frac{n_2^{ij}-\delta^{ij}}{r_2 S} - \frac{(n_{12}^i + n_2^i)(n_{12}^j + n_2^j)}{S^2} \ , \tag{5.12b} $$

$$ {}_ig_j \equiv \partial_{1i}\partial_{2j} g = \frac{n_{12}^{ij}-\delta^{ij}}{r_{12}S} + \frac{(n_{12}^i - n_1^i)(n_{12}^j + n_2^j)}{S^2} \ . \tag{5.12c} $$

Contracting with the Kronecker δ^{ij}, the relations (5.12) become ${}_{ii}g = \Delta_1 g = 1/r_1 r_{12}$, $g_{ii} = \Delta_2 g = 1/r_2 r_{12}$, and, more interestingly,

$$ {}_ig_i = \frac{1}{2}\left(\frac{1}{r_1 r_2} - \frac{1}{r_2 r_{12}} - \frac{1}{r_1 r_{12}}\right) \ . \tag{5.13} $$

This simple result is a straightforward consequence of the helpful formulas

$$ \frac{1+(n_1n_2)}{S} = \frac{r_1+r_2-r_{12}}{2r_1r_2} \ , \tag{5.14a} $$

$$ \frac{1-(n_1n_{12})}{S} = \frac{r_1+r_{12}-r_2}{2r_1r_{12}} \ , \tag{5.14b} $$

$$ \frac{1+(n_2n_{12})}{S} = \frac{r_2+r_{12}-r_1}{2r_2r_{12}} \ . \tag{5.14c} $$

Finally, we find the values of g and its derivatives at the location of body 1 (say) according to the Hadamard regularization (3.6): $(g)_1 = \ln(2r_{12})$, together with

$$ ({}_ig)_1 = \frac{n_{12}^i}{2r_{12}} \ ; \qquad (g_i)_1 = -\frac{n_{12}^i}{r_{12}} \ , \tag{5.15a} $$

$$ ({}_ig_j)_1 = \frac{-\delta^{ij}+2n_{12}^{ij}}{2r_{12}^2} \ ; \quad ({}_ig_i)_1 = -\frac{1}{2r_{12}^2} \ , \tag{5.15b} $$

$$ ({}_{ij}g)_1 = \frac{\delta^{ij}-3n_{12}^{ij}}{4r_{12}^2} \ ; \qquad (g_{ij})_1 = \frac{\delta^{ij}-2n_{12}^{ij}}{r_{12}^2} \ . \tag{5.15c} $$

These formulas are extensively used when getting the potentials at 1 (see Appendix B).

VI. THE NON-COMPACT POTENTIAL

The so-called non-compact potential is defined by (2.10) as the retarded integral of a source composed of the product of a V-type potential and $\hat{W}_{ij}^{(\partial V\partial V)}$, the latter potential being itself given as the retarded integral of a source made of a quadratic product of V's. Due to this purely cubic structure, one would expect *a priori* that the computation of the

non-compact term represents a non-trivial task, and even that it is not at all guaranteed that this term can be expressible with the help of simple algebraic functions (algebraically closed-form). Rather surprisingly, the NC potential turns out to accept an algebraically closed-form up to 0.5PN order. As a result, one can find its explicit expression, valid for any source point $\mathbf{x}$ (the value when the source point sits on a particle $\mathbf{y}_{1,2}$ following from the regularization process). To Newtonian order the closed-form expression of the NC term has already been obtained in [41,42] by combining some technical results derived earlier in [54,31,55]. We shall present here a slightly different but totally equivalent form of $\hat{X}^{(\mathrm{NC})}$ at Newtonian order, and add to this the 0.5PN correction. Very likely the 1PN and higher corrections in $\hat{X}^{(\mathrm{NC})}$ do not admit any algebraically closed-form all over space-time, but the regularized values at the location of the two bodies can probably be carried out explicitly (these values are needed when investigating the equations of motion).

To 0.5PN order the non-compact potential reads as

$$\begin{aligned}\hat{X}^{(\mathrm{NC})} &= \Box_R^{-1}\left\{\hat{W}_{ij}^{(\partial V\partial V)}\partial^2_{ij}V\right\}\\ &= \Delta^{-1}\left\{\hat{W}_{ij}^{(\partial V\partial V)}\partial^2_{ij}V\right\} + \frac{1}{4\pi c}\frac{d}{dt}\int d^3\mathbf{x}\left\{\hat{W}_{ij}^{(\partial V\partial V)}\partial^2_{ij}V\right\} + O(2)\ ,\end{aligned}\tag{6.1}$$

where to this order V can be replaced by (3.8a) and $\hat{W}_{ij}^{(\partial V\partial V)}$ by (5.10). The cubic source is easily obtained thanks to (3.8a) and (5.10). Hence we arrive at

$$\begin{aligned}\hat{W}_{ij}^{(\partial V\partial V)}\partial^2_{ij}V &= \frac{G^3m_1^3}{2}\left[\frac{1}{r_1^5}+\frac{\pi}{r_1^2}\delta(\mathbf{x}-\mathbf{y}_1)\right]\\ &+ G^3m_1^2m_2\left\{\frac{\pi}{2}\frac{\delta(\mathbf{x}-\mathbf{y}_2)}{r_1^2} - \frac{1}{8}\partial^2_{ij}\left(\frac{1}{r_2}\right)\partial^2_{ij}\ln r_1 - 2\partial^2_{ij}\left(\frac{1}{r_1}\right){}_ig_j\right\}\\ &+ \frac{G^3m_1^2m_2}{cr_{12}^2}\left[n_{12}^iv_{12}^j - \frac{1}{2}\left(3n_{12}^{ij}-\delta^{ij}\right)(n_{12}v_{12})\right]\partial^2_{1ij}\left(\frac{2}{r_1}\right)\\ &+ O(2) + 1\leftrightarrow 2\ .\end{aligned}\tag{6.2}$$

We compute the Poisson integral of (6.2). The (ill-defined) distributional term in the self part ($\propto m_1^3$) of (6.2) is treated unambiguously using the rule (3.7) and does not contribute to the Poisson integral. On the contrary the distributional term in the interaction part ($\propto m_1^2m_2$) is well-defined and gives a net contribution. Easy terms are obtained from the

facts that $\Delta(1/r_1^3) = 6/r_1^5$ and $\Delta(r_1) = 2/r_1$. The difficult point is to find the solutions of the two Poisson equations

$$\Delta K_1 = 2\,\partial^2_{ij}\left(\frac{1}{r_2}\right)\partial^2_{ij}\ln r_1\ , \tag{6.3a}$$

$$\Delta H_1 = 2\,\partial^2_{ij}\left(\frac{1}{r_1}\right){}_ig_j\ . \tag{6.3b}$$

We use the same notation as in [42] except that we add a subscript 1 to distinguish a function from its image obtained by the exchange of bodies 1 and 2. Remarkably, the solutions of (6.3) can be written down everywhere in space-time under explicit form [41,42,56]:

$$\begin{aligned} K_1 &= \left(\frac{1}{2}\Delta - \Delta_1\right)\left[\frac{\ln r_1}{r_2}\right] \\ &\quad + \frac{1}{2}\Delta_2\left[\frac{\ln r_{12}}{r_2}\right] + \frac{r_2}{2r_{12}^2 r_1^2} + \frac{1}{r_{12}^2 r_2}\ , \end{aligned} \tag{6.4a}$$

$$\begin{aligned} H_1 &= \frac{1}{2}\Delta_1\left[\frac{g}{r_1} + \frac{\ln r_1}{r_{12}} - \Delta_1\left(\frac{r_1 + r_{12}}{2}g\right)\right] \\ &\quad + \partial_i\partial_{2i}\left[\frac{\ln r_{12}}{r_1} + \frac{\ln r_1}{2r_{12}}\right] - \frac{1}{r_1}\partial_{2i}[(\partial_i g)_1] - \frac{r_2}{2r_1^2 r_{12}^2}\ . \end{aligned} \tag{6.4b}$$

They are equivalent to the expressions given by the equations (3.48)-(3.49) in [42]. By expanding all derivatives, we come to the completely developed forms

$$K_1 = -\frac{1}{r_2^3} + \frac{1}{r_2 r_{12}^2} - \frac{1}{r_1^2 r_2} + \frac{r_2}{2r_1^2 r_{12}^2} + \frac{r_{12}^2}{2r_1^2 r_2^3} + \frac{r_1^2}{2r_2^3 r_{12}^2}\ , \tag{6.5a}$$

$$H_1 = -\frac{1}{2r_1^3} - \frac{1}{4r_{12}^3} - \frac{1}{4r_1^2 r_{12}} - \frac{r_2}{2r_1^2 r_{12}^2} + \frac{r_2}{2r_1^3 r_{12}} + \frac{3r_2^2}{4r_1^2 r_{12}^3} + \frac{r_2^2}{2r_1^3 r_{12}^2} - \frac{r_2^3}{2r_1^3 r_{12}^3} \tag{6.5b}$$

(K_2 and H_2 are obtained by exchanging r_1 and r_2 in the right-hand-sides). With the solutions (6.4)-(6.5), we control the NC potential at the Newtonian approximation.

Next, we compute the spatial integral of (6.2) entering the 0.5PN correction in the NC potential. We must evaluate essentially the spatial integrals of the two source terms in the right-hand-sides of (6.3). We proceed like for the integral of $1/r_1r_2$ in (5.7)-(5.8). Namely, we integrate over a ball of constant radius $\mathcal{R}$, and use the function K_1 to transform the volume integral into a surface integral over the sphere $r = \mathcal{R}$:

$$2\int_{|\mathbf{x}|\leq\mathcal{R}} d^3\mathbf{x}\ \partial^2_{ij}\left(\frac{1}{r_2}\right)\partial^2_{ij}\ln r_1 = \int_{|\mathbf{x}|\leq\mathcal{R}} d^3\mathbf{x}\ \Delta K_1 = \int_{r=\mathcal{R}} d\Omega\ (r^2\partial_r K_1)\ . \tag{6.6}$$

From the developed expression of K_1 given by (6.5a), we get $K_1 = 2/rr_{12}^2 + O(1/r^2)$, which, when substituted into the surface integral in (6.6), yields

$$-\frac{1}{2\pi}\int_{|\mathbf{x}|\leq\mathcal{R}} d^3\mathbf{x}\;\partial^2_{ij}\left(\frac{1}{r_2}\right)\partial^2_{ij}\ln r_1 = \frac{2}{r_{12}^2} + O\left(\frac{1}{\mathcal{R}}\right)\;. \tag{6.7}$$

As we can see, the integral is finite in the limit $\mathcal{R}\to\infty$, with value

$$-\frac{1}{2\pi}\int d^3\mathbf{x}\;\partial^2_{ij}\left(\frac{1}{r_2}\right)\partial^2_{ij}\ln r_1 = \frac{2}{r_{12}^2}\;. \tag{6.8}$$

The same method applied to H_1 leads, since $H_1 = O(1/r^2)$, to

$$-\frac{1}{2\pi}\int d^3\mathbf{x}\;\partial^2_{ij}\left(\frac{1}{r_1}\right){}_ig_j = 0\;. \tag{6.9}$$

We must compute now the spatial integral of $1/r_1^5$ [see the first term in (6.2)]. It is clearly infinite because of the divergency at the bound $\mathbf{x}\to\mathbf{y}_1$. By integrating $1/r_1^5$ from $r_1 = \epsilon$ up to infinity, we obtain: $\int_{r_1\geq\epsilon} d^3\mathbf{x}/r_1^5 = 2\pi/\epsilon^2$, which is a pure constant [57], cancelled after applying the time-derivative in front of the $1/c$-term in (6.1). The second term in (6.1) is therefore

$$\frac{1}{4\pi c}\frac{d}{dt}\int d^3\mathbf{x}\left\{\hat{W}^{(\partial V\partial V)}_{ij}\partial^2_{ij}V\right\} = -\frac{G^3m_1^2m_2}{2cr_{12}^3}(n_{12}v_{12}) + O(2) + 1\leftrightarrow 2\;. \tag{6.10}$$

By summing the various contributions, we find the non-compact potential at 0.5PN order:

$$\begin{aligned}\hat{X}^{(\mathrm{NC})} &= \frac{G^3m_1^3}{12r_1^3} - G^3m_1^2m_2\left\{\frac{1}{8r_2r_{12}^2} + \frac{1}{16}K_1 + H_1\right\}\\ &+ \frac{G^3m_1^2m_2}{cr_{12}^2}\left[n_{12}^iv_{12}^j - \frac{1}{2}\left(3n_{12}^{ij} - \delta^{ij}\right)(n_{12}v_{12})\right]\partial^2_{1ij}r_1\\ &- \frac{G^3m_1^2m_2}{2cr_{12}^3}(n_{12}v_{12}) + O(2) + 1\leftrightarrow 2\;.\end{aligned} \tag{6.11}$$

Adding the other contributions in $\hat{X}$ we end up with the complete expression reported in the Appendix B.

Finally, we give the value of the non-compact potential (6.11) at the location of body 1. From the Hadamard recipe (3.6) we find for the functions $K_{1,2}$ and $H_{1,2}$ at point 1:

$$(K_1)_1 = \frac{2}{3r_{12}^3}\;;\quad (K_2)_1 = 0\;, \tag{6.12a}$$

$$(H_1)_1 = \frac{1}{3r_{12}^3}\;;\quad (H_2)_1 = -\frac{1}{r_{12}^3}\;, \tag{6.12b}$$

so that the non-compact potential at point 1 is

$$(\hat{X}^{(\mathrm{NC})})_1 = \frac{G^3 m_2}{r_{12}^3}\left[-\frac{1}{2}m_1^2 + m_1 m_2 + \frac{1}{12}m_2^2\right] + \frac{G^3 m_1 m_2}{2c r_{12}^3}(-m_1 + m_2)(n_{12}v_{12}) + O(2)\ . \tag{6.13}$$

VII. THE 2.5PN METRIC OF BINARY SYSTEMS

From the results of the previous sections, we are in the position to write down the 2.5PN harmonic-coordinate metric generated by two point-like particles as a function of the coordinate position $\mathbf{x}$ and a functional of the coordinate positions and velocities of the particles $\mathbf{y}_{1,2}(t), \mathbf{v}_{1,2}(t)$ (where t =const is the harmonic-coordinate slicing):

$$g_{\mu\nu}(\mathbf{x}, t) = g_{\mu\nu}[\mathbf{x}; \mathbf{y}_1(t), \mathbf{y}_2(t); \mathbf{v}_1(t), \mathbf{v}_2(t)]\ . \tag{7.1}$$

The metric is given by (2.5) in which we insert the expressions of the potentials as listed in Appendix B. After combining together identical terms we obtain [58]:

$$\begin{aligned}
g_{00} + 1 = \frac{2Gm_1}{c^2 r_1} &+ \frac{1}{c^4}\left[\frac{Gm_1}{r_1}\left(-(n_1v_1)^2 + 4v_1^2\right) - 2\frac{G^2m_1^2}{r_1^2}\right. \\
&\left. + G^2 m_1 m_2\left(-\frac{2}{r_1 r_2} - \frac{r_1}{2r_{12}^3} + \frac{r_1^2}{2r_2 r_{12}^3} - \frac{5}{2r_2 r_{12}}\right)\right] + \frac{4G^2 m_1 m_2}{3c^5 r_{12}^2}(n_{12}v_{12}) \\
&+ \frac{1}{c^6}\left[\frac{Gm_1}{r_1}\left(\frac{3}{4}(n_1v_1)^4 - 3(n_1v_1)^2 v_1^2 + 4v_1^4\right) + \frac{G^2m_1^2}{r_1^2}\left(3(n_1v_1)^2 - v_1^2\right) + 2\frac{G^3m_1^3}{r_1^3}\right. \\
&+ G^2 m_1 m_2\left(v_1^2\left(\frac{3r_1^3}{8r_{12}^5} - \frac{3r_1^2 r_2}{8r_{12}^5} - \frac{3r_1 r_2^2}{8r_{12}^5} + \frac{3r_2^3}{8r_{12}^5} - \frac{37r_1}{8r_{12}^3} + \frac{r_1^2}{r_2 r_{12}^3} + \frac{3r_2}{8r_{12}^3}\right.\right. \\
&\qquad \left. + \frac{2r_2^2}{r_1 r_{12}^3} + \frac{6}{r_1 r_{12}} - \frac{5}{r_2 r_{12}} - \frac{8r_{12}}{r_1 r_2 S} + \frac{16}{r_{12}S}\right) \\
&\quad + (v_1v_2)\left(\frac{8}{r_1 r_2} - \frac{3r_1^3}{4r_{12}^5} + \frac{3r_1^2 r_2}{4r_{12}^5} + \frac{13r_1}{4r_{12}^3} - \frac{2r_1^2}{r_2 r_{12}^3} - \frac{6}{r_1 r_{12}} - \frac{16}{r_1 S} - \frac{12}{r_{12}S}\right) \\
&\quad + (n_{12}v_1)^2\left(-\frac{15r_1^3}{8r_{12}^5} + \frac{15r_1^2 r_2}{8r_{12}^5} + \frac{15r_1 r_2^2}{8r_{12}^5} - \frac{15r_2^3}{8r_{12}^5} + \frac{57r_1}{8r_{12}^3} - \frac{3r_1^2}{4r_2 r_{12}^3} - \frac{33r_2}{8r_{12}^3}\right. \\
&\qquad \left. + \frac{7}{4r_2 r_{12}} - \frac{16}{S^2} - \frac{16}{r_{12}S}\right) \\
&\quad + (n_{12}v_1)(n_{12}v_2)\left(\frac{15r_1^3}{4r_{12}^5} - \frac{15r_1^2 r_2}{4r_{12}^5} - \frac{9r_1}{4r_{12}^3} + \frac{12}{S^2} + \frac{12}{r_{12}S}\right)
\end{aligned}$$

$$
\begin{aligned}
&+(n_1v_1)^2\left(\frac{2}{r_1r_2}-\frac{r_1}{4r_{12}^3}-\frac{3r_2^2}{4r_1r_{12}^3}+\frac{7}{4r_1r_{12}}-\frac{8}{S^2}-\frac{8}{r_1S}\right)\\
&+(n_1v_1)(n_1v_2)\left(\frac{r_1}{r_{12}^3}+\frac{16}{S^2}+\frac{16}{r_1S}\right)-(n_1v_2)^2\left(\frac{8}{S^2}+\frac{8}{r_1S}\right)\\
&+(n_{12}v_1)(n_1v_1)\left(-\frac{3r_1^2}{r_{12}^4}+\frac{3r_2^2}{2r_{12}^4}+\frac{3}{2r_{12}^2}+\frac{16}{S^2}\right)+\frac{16(n_1v_2)(n_2v_1)}{S^2}\\
&+(n_{12}v_2)(n_1v_1)\left(\frac{3r_1^2}{r_{12}^4}-\frac{3r_2^2}{2r_{12}^4}+\frac{13}{2r_{12}^2}-\frac{40}{S^2}\right)-\frac{12(n_1v_1)(n_2v_2)}{S^2}\\
&+(n_{12}v_1)(n_1v_2)\left(\frac{3r_1^2}{2r_{12}^4}+\frac{4}{r_{12}^2}+\frac{16}{S^2}\right)+(n_{12}v_2)(n_1v_2)\left(\frac{-3r_1^2}{2r_{12}^4}-\frac{3}{r_{12}^2}+\frac{16}{S^2}\right)\Bigg)\\
&+G^3m_1^2m_2\Bigg(\frac{4}{r_1^3}+\frac{1}{2r_2^3}+\frac{9}{2r_1^2r_2}-\frac{r_1^3}{4r_{12}^6}+\frac{3r_1^4}{16r_2r_{12}^6}-\frac{r_1^2r_2}{8r_{12}^6}+\frac{r_1r_2^2}{4r_{12}^6}-\frac{r_2^3}{16r_{12}^6}+\frac{5r_1}{4r_{12}^4}\\
&\quad-\frac{23r_1^2}{8r_2r_{12}^4}+\frac{43r_2}{8r_{12}^4}-\frac{5r_2^2}{2r_1r_{12}^4}-\frac{3}{r_{12}^3}+\frac{3r_1}{r_2r_{12}^3}+\frac{r_2}{r_1r_{12}^3}-\frac{5r_2^2}{r_1^2r_{12}^3}+\frac{4r_2^3}{r_1^3r_{12}^3}+\frac{3}{2r_1r_{12}^2}\\
&\quad-\frac{r_1^2}{4r_2^3r_{12}^2}+\frac{3}{16r_2r_{12}^2}+\frac{15r_2}{4r_1^2r_{12}^2}-\frac{4r_2^2}{r_1^3r_{12}^2}+\frac{5}{r_1^2r_{12}}+\frac{5}{r_1r_2r_{12}}-\frac{4r_2}{r_1^3r_{12}}-\frac{r_{12}^2}{4r_1^2r_2^3}\Bigg)\Bigg]\\
&+\frac{1}{c^7}\Bigg[G^2m_1m_2\Bigg((n_{12}v_{12})^2(n_1v_1)\left(-\frac{8r_1^3}{r_{12}^5}-\frac{16r_1}{r_{12}^3}\right)+(n_{12}v_{12})^2(n_1v_2)\left(\frac{8r_1^3}{r_{12}^5}+\frac{5r_1}{r_{12}^3}\right)\\
&\qquad+(n_{12}v_{12})^3\left(-\frac{7r_1^4}{2r_{12}^6}+\frac{7r_1^2r_2^2}{2r_{12}^6}-\frac{11r_1^2}{r_{12}^4}-\frac{37}{4r_{12}^2}\right)\\
&\qquad+(n_{12}v_1)(n_{12}v_{12})^2\left(\frac{20r_1^2}{r_{12}^4}-\frac{11}{2r_{12}^2}\right)-4(n_{12}v_{12})(n_1v_1)^2\frac{r_1^2}{r_{12}^4}\\
&\qquad+4(n_{12}v_{12})(n_1v_1)(n_1v_2)\frac{r_1^2}{r_{12}^4}+(n_{12}v_1)^2(n_1v_1)\frac{r_1}{r_{12}^3}\\
&\qquad+22(n_{12}v_1)(n_{12}v_{12})(n_1v_1)\frac{r_1}{r_{12}^3}-(n_{12}v_1)^2(n_1v_2)\frac{r_1}{r_{12}^3}\\
&\qquad+4(n_{12}v_1)(n_{12}v_{12})(n_1v_2)\frac{r_1}{r_{12}^3}+11(n_{12}v_1)^2(n_{12}v_{12})\frac{1}{2r_{12}^2}\\
&\qquad+(n_1v_2)v_{12}^2\left(-\frac{8r_1^3}{5r_{12}^5}-\frac{2r_1}{3r_{12}^3}\right)+(n_1v_1)v_{12}^2\left(\frac{8r_1^3}{5r_{12}^5}+\frac{11r_1}{3r_{12}^3}\right)\\
&\qquad-(n_{12}v_1)v_{12}^2\left(\frac{4r_1^2}{r_{12}^4}+\frac{5}{2r_{12}^2}\right)-(n_{12}v_{12})v_1^2\left(\frac{12r_1^2}{r_{12}^4}+\frac{5}{2r_{12}^2}\right)\\
&\qquad+(n_{12}v_{12})v_{12}^2\left(\frac{3r_1^4}{2r_{12}^6}-\frac{3r_1^2r_2^2}{2r_{12}^6}+\frac{7r_1^2}{r_{12}^4}+\frac{27}{4r_{12}^2}\right)\\
&\qquad-29(n_1v_1)v_1^2\frac{r_1}{3r_{12}^3}+(n_1v_2)v_1^2\frac{r_1}{r_{12}^3}+5(n_{12}v_1)v_1^2\frac{1}{r_{12}^2}\\
&\qquad+(n_{12}v_{12})(v_1v_2)\left(\frac{12r_1^2}{r_{12}^4}+\frac{3}{r_{12}^2}\right)+8(n_1v_1)(v_1v_2)\frac{r_1}{r_{12}^3}\\
&\qquad+2(n_1v_2)(v_1v_2)\frac{r_1}{3r_{12}^3}-5(n_{12}v_1)(v_1v_2)\frac{1}{r_{12}^2}\Bigg)
\end{aligned}
$$

$$+ G^3 m_1^2 m_2 \left((n_1 v_{12}) \left(-\frac{8r_1^3}{15r_{12}^6} + \frac{8r_1 r_2^2}{15r_{12}^6} - \frac{16r_1}{3r_{12}^4} + \frac{8}{r_{12}^3} - \frac{8r_2^2}{r_1^2 r_{12}^3} + \frac{8}{r_1^2 r_{12}} \right) \right.$$

$$+ (n_{12} v_1) \left(-\frac{4r_1^2}{3r_{12}^5} + \frac{4r_2^2}{3r_{12}^5} + \frac{20}{3r_{12}^3} \right) + 8(n_1 v_1) \frac{r_1}{3r_{12}^4}$$

$$+ (n_{12} v_{12}) \left(\frac{3}{r_1^3} - \frac{4r_1^2}{3r_{12}^5} + \frac{68r_2^2}{15r_{12}^5} + \frac{3r_1}{r_{12}^4} - \frac{6r_2^2}{r_1 r_{12}^4} + \frac{3r_2^4}{r_1^3 r_{12}^4} - \frac{76}{3r_{12}^3} \right.$$

$$\left. \left. \left. + \frac{2}{r_1 r_{12}^2} - \frac{6r_2^2}{r_1^3 r_{12}^2} \right) \right) \right] + O(8) + 1 \leftrightarrow 2 \,, \tag{7.2a}$$

$$g_{0i} = -4\frac{Gm_1}{c^3 r_1} v_1^i + \frac{1}{c^5} \left[n_1^i \left(-\frac{G^2 m_1^2}{r_1^2} (n_1 v_1) + \frac{G^2 m_1 m_2}{S^2} \left(-16(n_{12} v_1) + 12(n_{12} v_2) \right. \right. \right.$$

$$\left. \left. -16(n_2 v_1) + 12(n_2 v_2) \right) \right)$$

$$+ n_{12}^i G^2 m_1 m_2 \left(-6(n_{12} v_{12}) \frac{r_1}{r_{12}^3} - 4(n_1 v_1) \frac{1}{r_{12}^2} + 12(n_1 v_1) \frac{1}{S^2} \right.$$

$$\left. -16(n_1 v_2) \frac{1}{S^2} + 4(n_{12} v_1) \frac{1}{S} \left(\frac{1}{S} + \frac{1}{r_{12}} \right) \right)$$

$$+ v_1^i \left(\frac{Gm_1}{r_1} \left(2(n_1 v_1)^2 - 4v_1^2 \right) + \frac{G^2 m_1^2}{r_1^2} + G^2 m_1 m_2 \left(\frac{3r_1}{r_{12}^3} - \frac{2r_2}{r_{12}^3} \right) \right.$$

$$\left. \left. + G^2 m_1 m_2 \left(-\frac{r_2^2}{r_1 r_{12}^3} - \frac{3}{r_1 r_{12}} + \frac{8}{r_2 r_{12}} - \frac{4}{r_{12} S} \right) \right) \right]$$

$$+ \frac{1}{c^6} \left[n_{12}^i \left(G^2 m_1 m_2 \left(-10(n_{12} v_{12})^2 \frac{r_1^2}{r_{12}^4} - 12(n_{12} v_{12})(n_1 v_1) \frac{r_1}{r_{12}^3} + 2v_{12}^2 \frac{r_1^2}{r_{12}^4} - 4\frac{v_1^2}{r_{12}^2} \right) \right. \right.$$

$$\left. + G^3 m_1^2 m_2 \left(\frac{2r_1^2}{3r_{12}^5} - \frac{2r_2^2}{3r_{12}^5} - \frac{2}{r_{12}^3} \right) \right)$$

$$+ v_1^i \frac{G^2 m_1 m_2}{r_{12}^3} \left(\frac{16(n_1 v_{12}) r_1}{3} - 4(n_{12} v_2) r_{12} \right)$$

$$\left. + v_{12}^i \frac{G^2 m_1 m_2}{r_{12}^2} \left(-2(n_{12} v_1) + 6(n_{12} v_{12}) \frac{r_1^2}{r_{12}^2} \right) \right] + O(7) + 1 \leftrightarrow 2 \,, \tag{7.2b}$$

$$g_{ij} - \delta_{ij} = 2\frac{Gm_1}{c^2 r_1} \delta^{ij} + \frac{1}{c^4} \left[\delta^{ij} \left(-\frac{Gm_1}{r_1} (n_1 v_1)^2 + \frac{G^2 m_1^2}{r_1^2} \right. \right.$$

$$\left. + G^2 m_1 m_2 \left(\frac{2}{r_1 r_2} - \frac{r_1}{2r_{12}^3} + \frac{r_1^2}{2r_2 r_{12}^3} - \frac{5}{2r_1 r_{12}} + \frac{4}{r_{12} S} \right) \right)$$

$$+ 4\frac{Gm_1}{r_1} v_1^i v_1^j + \frac{G^2 m_1^2}{r_1^2} n_1^i n_1^j - 4G^2 m_1 m_2 n_{12}^i n_{12}^j \left(\frac{1}{S^2} + \frac{1}{r_{12} S} \right)$$

$$\left. + \frac{4G^2 m_1 m_2}{S^2} \left(n_1^{(i} n_2^{j)} + 2n_1^{(i} n_{12}^{j)} \right) \right]$$

$$+\frac{G^2m_1m_2}{c^5r_{12}^2}\left(-\frac{2}{3}(n_{12}v_{12})\delta^{ij}-6(n_{12}v_{12})n_{12}^in_{12}^j+8n_{12}^{(i}v_{12}^{j)}\right)+O(6)+1\leftrightarrow 2\ , \qquad (7.2c)$$

(where we recall $S = r_1 + r_2 + r_{12}$). The harmonic-coordinate conditions satisfied by (7.2) read, with this order of approximation: $\partial_\nu[(-g)^{1/2}g^{0\nu}] = O(7)$ and $\partial_\nu[(-g)^{1/2}g^{i\nu}] = O(6)$.

When applied to post-Newtonian initial conditions for the numerical evolution of two compact objects [1,2] (provided that the initial spatial numerical grid does not extend outside the binary's near zone [40]), the lengthy expressions (7.2) become a little simpler as the orbit can be considered as circular with a good approximation. In this case we have $(n_{12}v_1) = O(5) = (n_{12}v_2)$, the remainder $O(5)$ corresponding to radiation-reaction effects. Obviously, all the resulting $O(5)$'s can be neglected since they yield terms falling into the uncontrolled remainders of (7.2). If in addition we are working in a mass centered frame, then $\mathbf{y}_1 = X_2\mathbf{y}_{12} + O(4)$ and $\mathbf{y}_2 = -X_1\mathbf{y}_{12} + O(4)$, where $X_1 = m_1/m$, $X_2 = m_2/m$, $m = m_1 + m_2$. All the remainders $O(4)$ become negligible after insertion in (7.2). Thus, for instance: $v_1^2 = X_2^2v_{12}^2 + O(4)$, and

$$(n_1v_1) = X_2\frac{r}{r_1}(nv_{12}) + O(4)\ , \qquad (7.3a)$$

$$(n_1v_2) = -X_1\frac{r}{r_1}(nv_{12}) + O(4) \qquad (7.3b)$$

(plus the same formulas with 1 and 2 exchanged). We denote by $\mathbf{n} = \mathbf{x}/r$ and $r = |\mathbf{x}|$ the direction and the distance from the center of mass; r depends on the two individual distances $r_{1,2}$ through the relation

$$r^2 = X_1r_1^2 + X_2r_2^2 - X_1X_2r_{12}^2 + O(4)\ . \qquad (7.4)$$

The magnitude of the relative velocity is $v_{12} = r_{12}\ \omega_{2\mathrm{PN}} + O(6)$, where $\omega_{2\mathrm{PN}}$ denotes the orbital frequency of the circular motion at 2PN order, and is given by (8.6) below [59].

We provide also the values of the metric coefficients (7.2) computed at body 1 (since these might also be needed in the problem of binary coalescence), i.e.

$$(g_{\mu\nu})_1(t) = g_{\mu\nu}[\mathbf{y}_1(t);\mathbf{y}_1(t),\mathbf{y}_2(t);\mathbf{v}_1(t),\mathbf{v}_2(t)]\ , \qquad (7.5)$$

where the limit $\mathbf{x} \to \mathbf{y}_1$ is understood in the sense of (3.6). Directly from (7.2), or using the expressions of the potentials at 1 as given in Appendix B, we get:

$$
\begin{aligned}
(g_{00})_1 = & -1 + 2\frac{Gm_2}{c^2 r_{12}} + \frac{Gm_2}{c^4 r_{12}}\left(4v_2^2 - (n_{12}v_2)^2 - 3\frac{Gm_1}{r_{12}} - 2\frac{Gm_2}{r_{12}}\right) \\
& + \frac{8G^2 m_1 m_2}{3c^5 r_{12}^2}(n_{12}v_{12}) + \frac{Gm_2}{c^6 r_{12}}\left(\frac{3}{4}(n_{12}v_2)^4 - 3(n_{12}v_2)^2 v_2^2 + 4v_2^4\right) \\
& + \frac{G^2 m_1 m_2}{c^6 r_{12}^2}\left(-\frac{87}{4}(n_{12}v_1)^2 + \frac{47}{2}(n_{12}v_1)(n_{12}v_2) - \frac{55}{4}(n_{12}v_2)^2 + \frac{23}{4}v_1^2 - \frac{39}{2}(v_1v_2)\right) \\
& + \frac{47}{4}\frac{G^2 m_1 m_2}{c^6 r_{12}^2}v_2^2 + \frac{Gm_2}{c^6 r_{12}}\left\{\frac{Gm_2}{r_{12}}\left[3(n_{12}v_2)^2 - v_2^2\right] - \frac{G^2 m_1^2}{r_{12}^2} + \frac{17}{2}\frac{G^2 m_1 m_2}{r_{12}^2} + 2\frac{G^2 m_2^2}{r_{12}^2}\right\} \\
& + \frac{G^2 m_1 m_2}{c^7 r_{12}^2}\left\{-20(n_{12}v_1)^3 + 40(n_{12}v_1)^2(n_{12}v_2) - 36(n_{12}v_1)(n_{12}v_2)^2 + 16(n_{12}v_2)^3\right. \\
& \qquad + \frac{296}{15}(n_{12}v_1)v_1^2 - \frac{116}{15}(n_{12}v_2)v_1^2 - \frac{104}{5}(n_{12}v_1)(v_1v_2) + \frac{232}{15}(n_{12}v_2)(v_1v_2) \\
& \qquad + \frac{56}{15}(n_{12}v_1)v_2^2 - \frac{52}{5}(n_{12}v_2)v_2^2 + \frac{Gm_1}{r_{12}}\left(-\frac{64}{5}(n_{12}v_1) + \frac{104}{5}(n_{12}v_2)\right) \\
& \qquad \left. + \frac{Gm_2}{r_{12}}\left(-\frac{144}{5}(n_{12}v_1) + \frac{392}{15}(n_{12}v_2)\right)\right\} + O(8) \ , \qquad (7.6a)
\end{aligned}
$$

$$
\begin{aligned}
(g_{0i})_1 = & -4\frac{Gm_2}{c^3 r_{12}}v_2^i + \frac{Gm_2}{c^5 r_{12}}\left\{n_{12}^i\left[\frac{Gm_1}{r_{12}}\left(10(n_{12}v_1) + 2(n_{12}v_2)\right) - \frac{Gm_2}{r_{12}}(n_{12}v_2)\right]\right\} \\
& + \frac{Gm_2}{c^5 r_{12}}\left\{4\frac{Gm_1}{r_{12}}v_1^i + v_2^i\left(2(n_{12}v_2)^2 - 4v_2^2 - 2\frac{Gm_1}{r_{12}} + \frac{Gm_2}{r_{12}}\right)\right\} \\
& + \frac{G^2 m_1 m_2}{c^6 r_{12}^2}\left\{n_{12}^i\left(10(n_{12}v_1)^2 - 8(n_{12}v_1)(n_{12}v_2) - 2(n_{12}v_2)^2 - 6v_1^2\right.\right. \\
& \qquad \left. + 4(v_1v_2) + 2v_2^2 - \frac{8}{3}\frac{Gm_1}{r_{12}} + \frac{4}{3}\frac{Gm_2}{r_{12}}\right) \\
& \qquad \left. - 8(n_{12}v_1)v_1^i + v_2^i\left(\frac{20}{3}(n_{12}v_1) + \frac{4}{3}(n_{12}v_2)\right)\right\} + O(7) \ , \qquad (7.6b)
\end{aligned}
$$

$$
\begin{aligned}
(g_{ij})_1 = & \ \delta^{ij} + 2\frac{Gm_2}{c^2 r_{12}}\delta^{ij} + \frac{Gm_2}{c^4 r_{12}}\delta^{ij}\left(-(n_{12}v_2)^2 + \frac{Gm_1}{r_{12}} + \frac{Gm_2}{r_{12}}\right) \\
& + \frac{Gm_2}{c^4 r_{12}}\left\{n_{12}^{ij}\left(-8\frac{Gm_1}{r_{12}} + \frac{Gm_2}{r_{12}}\right) + 4v_2^{ij}\right\} - \frac{4G^2 m_1 m_2}{3c^5 r_{12}^2}\delta^{ij}(n_{12}v_{12}) \\
& + \frac{G^2 m_1 m_2}{c^5 r_{12}^2}\left\{-12n_{12}^{ij}(n_{12}v_{12}) + 16n_{12}^{(i}v_{12}^{j)}\right\} + O(6) \ . \qquad (7.6c)
\end{aligned}
$$

Drastic simplifications occur in the case where the orbit is circular.

VIII. THE 2.5PN EQUATIONS OF MOTION OF BINARY SYSTEMS

The motion of body 1 under the gravitational influence of body 2 is simply the geodesic motion taking place in the post-Newtonian space-time (7.2). Now we write the geodesic equation of body 1 in the Newtonian-like form

$$\frac{d\mathcal{P}_1^i}{dt} = \mathcal{F}_1^i \ . \tag{8.1}$$

The "linear momentum" vector and "gravitational force" (per unit of mass) are defined by

$$\mathcal{P}_1^i \equiv \left(\frac{v_1^\mu g_{i\mu}}{\sqrt{-g_{\rho\sigma}\frac{v_1^\rho v_1^\sigma}{c^2}}}\right)_1 \qquad ; \qquad \mathcal{F}_1^i \equiv \frac{1}{2}\left(\frac{v_1^\mu v_1^\nu \partial_i g_{\mu\nu}}{\sqrt{-g_{\rho\sigma}\frac{v_1^\rho v_1^\sigma}{c^2}}}\right)_1 \ . \tag{8.2}$$

The above quantities are computed using the regularization (3.6) which is a crucial ingredient of our point-mass model.

Inserting the 2.5PN metric (2.5) into (8.2), we obtain

$$\begin{aligned}
\mathcal{P}_1^i = v_1^i &+ \frac{1}{c^2}\left[-4(V_i)_1 + 3(V)_1 v_1^i + \frac{1}{2}v_1^2 v_1^i\right] \\
&+ \frac{1}{c^4}\left[-8(\hat{R}_i)_1 + \frac{9}{2}(V^2)_1 v_1^i + 4(\hat{W}_{ij})_1 v_1^j - 4(VV_i)_1\right. \\
&\left.+ \frac{7}{2}(V)_1 v_1^2 v_1^i - 2v_1^2 (V_i)_1 - 4v_1^i v_1^j (V_j)_1 + \frac{3}{8}v_1^i v_1^4\right] + O(6) \ ,
\end{aligned} \tag{8.3a}$$

$$\begin{aligned}
\mathcal{F}_1^i = (\partial_i V)_1 &+ \frac{1}{c^2}\left[-(V\partial_i V)_1 + \frac{3}{2}v_1^2(\partial_i V)_1 - 4v_1^j(\partial_i V_j)_1\right] \\
&+ \frac{1}{c^4}\left[4(\partial_i \hat{X})_1 + 8(V_j \partial_i V_j)_1 - 8v_1^j(\partial_i \hat{R}_j)_1 + \frac{9}{2}v_1^2 (V\partial_i V)_1\right. \\
&+ 2v_1^j v_1^k (\partial_i \hat{W}_{jk})_1 - 2v_1^2 v_1^j (\partial_i V_j)_1 + \frac{7}{8}v_1^4(\partial_i V)_1 + \frac{1}{2}(V^2\partial_i V)_1 \\
&\left.- 4v_1^j (V_j \partial_i V)_1 - 4v_1^j (V\partial_i V_j)_1\right] + O(6) \ .
\end{aligned} \tag{8.3b}$$

Next we replace into these expressions all the potentials and their gradients computed at point 1 as given in Appendix B (to this order the Hadamard partie finie is "distributive" [50]), and get both $\mathcal{P}_1^i$ and $\mathcal{F}_1^i$ in terms of the relative separation $y_{12}^i = r_{12} n_{12}^i$ and individual velocities $v_{1,2}^i$ [alternatively we can obtain $\mathcal{P}_1^i$ and $\mathcal{F}_1^i$ directly from (7.2)]. Then, we compute

the time-derivative of $\mathcal{P}_1^i$, and order-reduce all the resulting accelerations (which appear at orders 1PN or 2PN) by means of the 1.5PN equations of motion given by (3.11). After insertion in (8.1) and simplification, we end with the 2.5PN acceleration of body 1:

$$\begin{aligned}
\frac{dv_1^i}{dt} = & -\frac{Gm_2}{r_{12}^2}n_{12}^i + \frac{Gm_2}{r_{12}^2c^2}\Big\{v_{12}^i\left[4(n_{12}v_1) - 3(n_{12}v_2)\right] \\
& \quad +n_{12}^i\left[-v_1^2 - 2v_2^2 + 4(v_1v_2) + \frac{3}{2}(n_{12}v_2)^2 + 5\frac{Gm_1}{r_{12}} + 4\frac{Gm_2}{r_{12}}\right]\Big\} \\
& + \frac{Gm_2}{r_{12}^2c^4}n_{12}^i\Big\{\Big[-2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 + \frac{3}{2}v_1^2(n_{12}v_2)^2 + \frac{9}{2}v_2^2(n_{12}v_2)^2 \\
& \quad -6(v_1v_2)(n_{12}v_2)^2 - \frac{15}{8}(n_{12}v_2)^4\Big] \\
& \quad +\frac{Gm_1}{r_{12}}\Big[-\frac{15}{4}v_1^2 + \frac{5}{4}v_2^2 - \frac{5}{2}(v_1v_2) \\
& \quad +\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2\Big] \\
& \quad +\frac{Gm_2}{r_{12}}\left[4v_2^2 - 8(v_1v_2) + 2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2\right] \\
& \quad +\frac{G^2}{r_{12}^2}\left[-\frac{57}{4}m_1^2 - 9m_2^2 - \frac{69}{2}m_1m_2\right]\Big\} \\
& + \frac{Gm_2}{r_{12}^2c^4}v_{12}^i\Big\{v_1^2(n_{12}v_2) + 4v_2^2(n_{12}v_1) - 5v_2^2(n_{12}v_2) - 4(v_1v_2)(n_{12}v_1) \\
& \quad +4(v_1v_2)(n_{12}v_2) - 6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 \\
& \quad +\frac{Gm_1}{r_{12}}\left[-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2)\right] + \frac{Gm_2}{r_{12}}\left[-2(n_{12}v_1) - 2(n_{12}v_2)\right]\Big\} \\
& + \frac{4G^2m_1m_2}{5c^5r_{12}^3}\Big\{n_{12}^i(n_{12}v_{12})\left[-6\frac{Gm_1}{r_{12}} + \frac{52}{3}\frac{Gm_2}{r_{12}} + 3v_{12}^2\right] \\
& \quad +v_{12}^i\left[2\frac{Gm_1}{r_{12}} - 8\frac{Gm_2}{r_{12}} - v_{12}^2\right]\Big\} + O(6)\ .
\end{aligned} \tag{8.4}$$

We find perfect agreement with the Damour-Deruelle [4–7] equations of motion. To emphasize the strength of this agreement we recall that the method employed in the present paper differs in many respects from the one originally used in [4–7] (see the discussion in the introduction). In the case of circular orbits, the equations reduce to

$$\frac{dv_{12}^i}{dt} = -\omega_{2\mathrm{PN}}^2 y_{12}^i - \frac{32G^3m^3\nu}{5c^5r_{12}^4}v_{12}^i + O(6)\ . \tag{8.5}$$

The second term represents the standard damping force, while the orbital frequency $\omega_{2\mathrm{PN}}$ is the frequency of the exact circular motion at 2PN order, related to the harmonic-coordinate

separation r_{12} by

$$\omega^2_{2\mathrm{PN}} \equiv \frac{Gm}{r^3_{12}} \left[1 + (-3+\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2\right] . \tag{8.6}$$

Our notation is: $m = m_1 + m_2$, $\nu = m_1 m_2/m^2$ ($= X_1 X_2$ in the notation of the previous section) and $\gamma = Gm/r_{12}c^2$.

APPENDIX A: DERIVATION OF THE 2.5PN FLUID METRIC

The derivation follows almost immediately from the results established in the section II.A of [39]. The Einstein field equations in harmonic coordinates are written as

$$\partial_\nu h^{\mu\nu} = 0 \ , \tag{A1a}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(h) \ , \tag{A1b}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat d'Alembertian operator [$\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$], $h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$, and $\Lambda^{\mu\nu}$ denotes the gravitational source term which is at least quadratic in h and its space-time derivatives (see [39] for the expressions of the quadratic and cubic parts of $\Lambda^{\mu\nu}$). From [39,46], we have, to order 1PN,

$$h^{00} = -\frac{4}{c^2}V - \frac{2}{c^4}\left(\hat{W}_{kk} + 4V^2\right) + O(6) \ , \tag{A2a}$$

$$h^{0i} = -\frac{4}{c^3}V_i + O(5) \ , \tag{A2b}$$

$$h^{ij} = -\frac{4}{c^4}\left[\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk}\right] + O(6) \ . \tag{A2c}$$

Substituting the 1PN metric into the right-hand-side of the field equation we get

$$|g| = 1 + \frac{4}{c^2}V + \frac{4}{c^4}\left(\hat{W}_{kk} + 2V^2\right) + O(6) \ , \tag{A3}$$

together with the gravitational source term (equations (2.12) in [39])

$$\Lambda^{00} = -\frac{14}{c^4}\partial_i V \partial_i V + \frac{16}{c^6}\Big\{ -V\partial_t^2 V - 2V_i\partial_t\partial_i V + \frac{5}{8}(\partial_t V)^2$$
$$+ \frac{1}{2}\partial_i V_j(\partial_i V_j + 3\partial_j V_i) + \partial_i V \partial_t V_i - \frac{7}{2}V\partial_i V\partial_i V$$

$$- \left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk}\right)\partial^2_{ij}V - \partial_i V\partial_i\hat{W}_{kk}\Big\} + O(8)\ , \tag{A4a}$$

$$\Lambda^{0i} = \frac{16}{c^5}\left\{\partial_j V(\partial_i V_j - \partial_j V_i) + \frac{3}{4}\partial_t V\partial_i V\right\} + O(7)\ , \tag{A4b}$$

$$\begin{aligned}\Lambda^{ij} = &\frac{4}{c^4}\left\{\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right\}\\ &+ \frac{16}{c^6}\Big\{2\partial_{(i}V\partial_t V_{j)} - \partial_i V_k\partial_j V_k - \partial_k V_i\partial_k V_j + 2\partial_{(i}V_k\partial_k V_{j)}\\ &- \frac{3}{8}\delta_{ij}(\partial_t V)^2 - \delta_{ij}\partial_k V\partial_t V_k + \frac{1}{2}\delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k)\Big\} + O(8)\ .\end{aligned} \tag{A4c}$$

These are the needed equations, which lead, by application of the retarded integral on the right-hand-side of the field equations, to

$$\frac{h^{00}+h^{ii}}{2} = -\frac{2}{c^2}V - \frac{4}{c^4}V^2 - \frac{8}{c^6}\left[\hat{X} + \frac{1}{2}V\hat{W}_{ii} + \frac{2V^3}{3}\right] + O(8)\ , \tag{A5a}$$

$$h^{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\left[\hat{R}_i + VV_i\right] + O(7)\ , \tag{A5b}$$

$$h^{ij} = -\frac{4}{c^4}\left[\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk}\right] + O(6) \tag{A5c}$$

(the potentials are defined in the text). From this, we deduce the components of the covariant metric $g_{\mu\nu}$ and find the result (2.5). It has been shown in section III of [39] that the post-Newtonian metric matches in the external near zone to a solution extending up to the radiative zone.

APPENDIX B: COMPLETE RESULTS FOR THE POTENTIALS

We give first all the relevant potentials (valid all-over space-time) which are used in the obtention of the 2.5PN metric (7.2):

$$\begin{aligned}V = &\frac{Gm_1}{r_1} + \frac{Gm_1}{c^2}\left(-\frac{(n_1v_1)^2}{2r_1} + \frac{2v_1^2}{r_1} + Gm_2\left(-\frac{r_1}{4r_{12}^3} - \frac{5}{4r_1r_{12}} + \frac{r_2^2}{4r_1r_{12}^3}\right)\right)\\ &+ \frac{2G^2m_1m_2(n_{12}v_{12})}{3c^3r_{12}^2} + \frac{Gm_1}{c^4r_1}\left(\frac{3(n_1v_1)^4}{8} - \frac{3(n_1v_1)^2v_1^2}{2} + 2v_1^4\right)\\ &+ \frac{G^2m_1m_2}{c^4}\Bigg\{v_1^2\left(\frac{3r_1^3}{16r_{12}^5} - \frac{37r_1}{16r_{12}^3} - \frac{1}{r_1r_{12}} - \frac{3r_1r_2^2}{16r_{12}^5} + \frac{r_2^2}{r_1r_{12}^3}\right)\\ &\qquad + v_2^2\left(\frac{3r_1^3}{16r_{12}^5} + \frac{3r_1}{16r_{12}^3} + \frac{3}{2r_1r_{12}} - \frac{3r_1r_2^2}{16r_{12}^5} + \frac{r_2^2}{2r_1r_{12}^3}\right)\end{aligned}$$

$$
\begin{aligned}
&\qquad +(v_1v_2)\left(-\frac{3r_1^3}{8r_{12}^5}+\frac{13r_1}{8r_{12}^3}-\frac{3}{r_1r_{12}}+\frac{3r_1r_2^2}{8r_{12}^5}-\frac{r_2^2}{r_1r_{12}^3}\right)\\
&\qquad +(n_{12}v_1)^2\left(-\frac{15r_1^3}{16r_{12}^5}+\frac{57r_1}{16r_{12}^3}+\frac{15r_1r_2^2}{16r_{12}^5}\right)\\
&\qquad +(n_{12}v_2)^2\left(-\frac{15r_1^3}{16r_{12}^5}-\frac{33r_1}{16r_{12}^3}+\frac{7}{8r_1r_{12}}+\frac{15r_1r_2^2}{16r_{12}^5}-\frac{3r_2^2}{8r_1r_{12}^3}\right)\\
&\qquad +(n_{12}v_1)(n_{12}v_2)\left(\frac{15r_1^3}{8r_{12}^5}-\frac{9r_1}{8r_{12}^3}-\frac{15r_1r_2^2}{8r_{12}^5}\right)\\
&\qquad +(n_1v_1)(n_{12}v_1)\left(-\frac{3r_1^2}{2r_{12}^4}+\frac{3}{4r_{12}^2}+\frac{3r_2^2}{4r_{12}^4}\right)+(n_1v_2)(n_{12}v_1)\left(\frac{3r_1^2}{4r_{12}^4}+\frac{2}{r_{12}^2}\right)\\
&\qquad +(n_1v_1)(n_{12}v_2)\left(\frac{3r_1^2}{2r_{12}^4}+\frac{13}{4r_{12}^2}-\frac{3r_2^2}{4r_{12}^4}\right)+(n_1v_2)(n_{12}v_2)\left(-\frac{3r_1^2}{4r_{12}^4}-\frac{3}{2r_{12}^2}\right)\\
&\qquad +(n_1v_1)^2\left(-\frac{r_1}{8r_{12}^3}+\frac{7}{8r_1r_{12}}-\frac{3r_2^2}{8r_1r_{12}^3}\right)+\frac{(n_1v_1)(n_1v_2)r_1}{2r_{12}^3}\Bigg\}\\
&+\frac{G^3m_1^2m_2}{c^4}\left(-\frac{r_1^3}{8r_{12}^6}+\frac{5r_1}{8r_{12}^4}+\frac{3}{4r_1r_{12}^2}+\frac{r_1r_2^2}{8r_{12}^6}-\frac{5r_2^2}{4r_1r_{12}^4}\right)\\
&+\frac{G^3m_1m_2^2}{c^4}\left(-\frac{r_1^3}{32r_{12}^6}+\frac{43r_1}{16r_{12}^4}+\frac{91}{32r_1r_{12}^2}-\frac{r_1r_2^2}{16r_{12}^6}-\frac{23r_2^2}{16r_1r_{12}^4}+\frac{3r_2^4}{32r_1r_{12}^6}\right)\\
&+\frac{G^2m_1m_2}{c^5}\Bigg\{(n_{12}v_{12})^3\left(-\frac{7r_1^4}{4r_{12}^6}+\frac{3r_1^2}{2r_{12}^4}-\frac{11}{8r_{12}^2}+\frac{7r_1^2r_2^2}{4r_{12}^6}\right)\\
&\qquad +(n_1v_{12})(n_{12}v_{12})^2\left(-\frac{2r_1^3}{r_{12}^5}-\frac{5r_1}{r_{12}^3}+\frac{2r_1r_2^2}{r_{12}^5}\right)-(n_1v_{12})^2(n_{12}v_{12})\frac{r_1^2}{r_{12}^4}\\
&\qquad +(n_{12}v_1)(n_{12}v_{12})^2\left(\frac{5r_1^2}{r_{12}^4}+\frac{5}{r_{12}^2}-\frac{5r_2^2}{r_{12}^4}\right)+(n_1v_{12})(n_{12}v_1)(n_{12}v_{12})\frac{6r_1}{r_{12}^3}\\
&\qquad +(n_1v_1)(n_{12}v_{12})^2\frac{2r_1}{r_{12}^3}-\frac{(n_{12}v_1)^2(n_{12}v_{12})}{r_{12}^2}-(n_1v_{12})v_1^2\frac{8r_1}{3r_{12}^3}+(n_{12}v_1)v_1^2\frac{8}{3r_{12}^2}\\
&\qquad +(n_{12}v_{12})v_1^2\left(-\frac{3r_1^2}{r_{12}^4}-\frac{8}{3r_{12}^2}+\frac{3r_2^2}{r_{12}^4}\right)+(n_{12}v_{12})(v_1v_2)\frac{7}{3r_{12}^2}\\
&\qquad +(n_1v_{12})(v_1v_2)\frac{8r_1}{3r_{12}^3}-(n_{12}v_1)(v_1v_2)\frac{8}{3r_{12}^2}-(n_1v_1)v_{12}^2\frac{2r_1}{3r_{12}^3}\\
&\qquad +(n_{12}v_{12})v_{12}^2\left(\frac{3r_1^4}{4r_{12}^6}-\frac{9r_1^2}{10r_{12}^4}-\frac{9}{8r_{12}^2}-\frac{3r_1^2r_2^2}{4r_{12}^6}\right)\\
&\qquad +(n_1v_{12})v_{12}^2\left(\frac{2r_1^3}{5r_{12}^5}+\frac{5r_1}{3r_{12}^3}-\frac{2r_1r_2^2}{5r_{12}^5}\right)+(n_{12}v_1)v_{12}^2\left(-\frac{r_1^2}{r_{12}^4}-\frac{5}{3r_{12}^2}+\frac{r_2^2}{r_{12}^4}\right)\Bigg\}\\
&+\frac{G^3m_1^2m_2}{c^5}\Bigg\{(n_1v_{12})\left(-\frac{4r_1^3}{15r_{12}^6}-\frac{8r_1}{3r_{12}^4}+\frac{4r_1r_2^2}{15r_{12}^6}\right)+(n_{12}v_1)\left(-\frac{2r_1^2}{3r_{12}^5}+\frac{10}{3r_{12}^3}+\frac{2r_2^2}{3r_{12}^5}\right)\\
&\qquad +(n_1v_1)\frac{4r_1}{3r_{12}^4}+(n_{12}v_{12})\left(-\frac{2r_1^2}{3r_{12}^5}-\frac{20}{3r_{12}^3}+\frac{8}{3r_1r_{12}^2}+\frac{34r_2^2}{15r_{12}^5}\right)\Bigg\}\\
&+O(6)+1\leftrightarrow 2\ , \qquad\qquad \text{(B1a)}
\end{aligned}
$$

$$
\begin{aligned}
V_i =& \frac{Gm_1 v_1^i}{r_1} + n_{12}^i \frac{G^2 m_1 m_2}{c^2 r_{12}^2}\left((n_1v_1) + \frac{3(n_{12}v_{12})r_1}{2r_{12}}\right)\\
&+ \frac{v_1^i}{c^2}\left\{\frac{Gm_1}{r_1}\left(-\frac{(n_1v_1)^2}{2} + v_1^2\right) + G^2 m_1 m_2\left(-\frac{3r_1}{4r_{12}^3} + \frac{r_2^2}{4r_1 r_{12}^3} - \frac{5}{4r_1 r_{12}}\right)\right\} + v_2^i \frac{G^2 m_1 m_2 r_1}{2c^2 r_{12}^3}\\
&+ \frac{n_{12}^i}{c^3} G^2 m_1 m_2 \left[(n_{12}v_{12})^2 \frac{5r_1^2}{2r_{12}^4} + (n_{12}v_{12})(n_1v_{12})\frac{3r_1}{2r_{12}^3} - v_{12}^2 \frac{r_1^2}{2r_{12}^4} + \frac{v_1^2}{2r_{12}^2}\right.\\
&\qquad\left. + Gm_1\left(-\frac{r_1^2}{6r_{12}^5} + \frac{r_2^2}{6r_{12}^5} + \frac{1}{2r_{12}^3}\right)\right] - \frac{G^2 m_1 m_2 v_1^i}{3c^3 r_{12}^2}\\
&+ \frac{G^2 m_1 m_2 v_{12}^i}{c^3 r_{12}^2}\left(-\frac{3r_1^2}{2r_{12}^4}(n_{12}v_{12}) - \frac{2r_1}{3r_{12}^3}(n_1v_{12}) + \frac{(n_{12}v_1)}{2r_{12}^2}\right) + O(4) + 1 \leftrightarrow 2\ , \qquad \text{(B1b)}\\
\hat{W}_{ij} =& \delta^{ij}\left(-\frac{Gm_1 v_1^2}{r_1} - \frac{G^2 m_1^2}{4r_1^2} + \frac{G^2 m_1 m_2}{r_{12} S}\right) + \frac{Gm_1 v_1^i v_1^j}{r_1} + \frac{G^2 m_1^2 n_1^i n_1^j}{4r_1^2}\\
&+ G^2 m_1 m_2\left\{\frac{1}{S^2}\left(n_1^{(i}n_2^{j)} + 2n_1^{(i}n_{12}^{j)}\right) - n_{12}^i n_{12}^j\left(\frac{1}{S^2} + \frac{1}{r_{12} S}\right)\right\}\\
&+ \frac{G^2 m_1 m_2}{c r_{12}^2}\left(-\frac{(n_{12}v_{12})}{2}\delta^{ij} - \frac{3(n_{12}v_{12})}{2} n_{12}^i n_{12}^j + 2n_{12}^{(i}v_{12}^{j)}\right)\\
&+ O(2) + 1 \leftrightarrow 2\ , \qquad \text{(B1c)}\\
\hat{R}_i =& G^2 m_1 m_2 n_{12}^i\left\{-\frac{(n_{12}v_1)}{2S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right) - \frac{2(n_2v_1)}{S^2} + \frac{3(n_2v_2)}{2S^2}\right\}\\
&+ n_1^i\left\{\frac{G^2 m_1^2 (n_1v_1)}{8r_1^2} + \frac{G^2 m_1 m_2}{S^2}\left(2(n_{12}v_1) - \frac{3(n_{12}v_2)}{2} + 2(n_2v_1) - \frac{3(n_2v_2)}{2}\right)\right\}\\
&+ v_1^i\left\{-\frac{G^2 m_1^2}{8r_1^2} + G^2 m_1 m_2\left(\frac{1}{r_1 r_{12}} + \frac{1}{2r_{12} S}\right)\right\} - v_2^i \frac{G^2 m_1 m_2}{r_1 r_{12}} + v_1^i \frac{G^2 m_1 m_2 (n_{12}v_1)}{2c r_{12}^2}\\
&+ n_{12}^i \frac{G^2 m_1 m_2}{c r_{12}^2}\left(-\frac{3(n_{12}v_1)^2}{4} + \frac{v_1^2}{4}\right) + O(2) + 1 \leftrightarrow 2\ , \qquad \text{(B1d)}\\
\hat{X} =& \frac{G^2 m_1^2}{8r_1^2}\left((n_1v_1)^2 - v_1^2\right) + G^2 m_1 m_2 v_1^2\left(\frac{1}{r_1 r_{12}} + \frac{1}{r_1 S} + \frac{1}{r_{12} S}\right)\\
&+ G^2 m_1 m_2\left\{v_2^2\left(-\frac{1}{r_1 r_{12}} + \frac{1}{r_1 S} + \frac{1}{r_{12} S}\right) - \frac{(v_1v_2)}{S}\left(\frac{2}{r_1} + \frac{3}{2r_{12}}\right) - \frac{(n_{12}v_1)^2}{S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right)\right.\\
&\qquad - \frac{(n_{12}v_2)^2}{S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right) + \frac{3(n_{12}v_1)(n_{12}v_2)}{2S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right) + \frac{2(n_{12}v_1)(n_1v_1)}{S^2}\\
&\qquad - \frac{5(n_{12}v_2)(n_1v_1)}{S^2} - \frac{(n_1v_1)^2}{S}\left(\frac{1}{S} + \frac{1}{r_1}\right) + \frac{2(n_{12}v_2)(n_1v_2)}{S^2}\\
&\qquad + \frac{2(n_1v_1)(n_1v_2)}{S}\left(\frac{1}{S} + \frac{1}{r_1}\right) - \frac{(n_1v_2)^2}{S}\left(\frac{1}{S} + \frac{1}{r_1}\right) - \frac{2(n_{12}v_2)(n_2v_1)}{S^2}\\
&\qquad\left. + \frac{2(n_1v_2)(n_2v_1)}{S^2} - \frac{3(n_1v_1)(n_2v_2)}{2S^2}\right\} + \frac{G^3 m_1^3}{12r_1^3}\\
&+ G^3 m_1^2 m_2\left(\frac{1}{2r_1^3} + \frac{1}{16r_2^3} + \frac{1}{16r_1^2 r_2} - \frac{r_2^2}{2r_1^2 r_{12}^3} + \frac{r_2^3}{2r_1^3 r_{12}^3} - \frac{r_1^2}{32r_2^3 r_{12}^2} - \frac{3}{16r_2 r_{12}^2} + \frac{15r_2}{32r_1^2 r_{12}^2}\right.
\end{aligned}
$$

$$
-\frac{r_2^2}{2r_1^3 r_{12}^2} - \frac{r_2}{2r_1^3 r_{12}} - \frac{r_{12}^2}{32 r_1^2 r_2^3}\Bigg) + G^3 m_1 m_2^2 \left(-\frac{1}{2r_{12}^3} + \frac{r_2}{2r_1 r_{12}^3} - \frac{1}{2r_1 r_{12}^2}\right)
$$

$$
+ \frac{G^2 m_1 m_2}{c r_{12}^2}\Bigg(-\frac{3(n_{12}v_1)^2(n_{12}v_{12})}{4} + \frac{3(n_{12}v_1)(n_{12}v_{12})^2}{4} - \frac{3(n_{12}v_{12})^3}{2} - \frac{(n_{12}v_1)v_{12}^2}{4}
$$

$$
+\frac{3(n_{12}v_{12})v_{12}^2}{2} + \frac{(n_{12}v_1)v_1^2}{2} - \frac{(n_{12}v_{12})v_1^2}{4} - \frac{(n_{12}v_1)(v_1v_2)}{2} + \frac{(n_{12}v_{12})(v_1v_2)}{2}\Bigg)
$$

$$
+ \frac{G^3 m_1^2 m_2}{c}\Bigg\{(n_{12}v_{12})\left(\frac{3}{8r_1^3} + \frac{3r_1}{8r_{12}^4} - \frac{3r_2^2}{4r_1 r_{12}^4} + \frac{3r_2^4}{8r_1^3 r_{12}^4} - \frac{3}{2r_{12}^3} + \frac{1}{4r_1 r_{12}^2} - \frac{3r_2^2}{4r_1^3 r_{12}^2}\right)
$$

$$
+(n_1v_{12})\left(\frac{1}{r_{12}^3} - \frac{r_2^2}{r_1^2 r_{12}^3} + \frac{1}{r_1^2 r_{12}}\right)\Bigg\} + O(2) + 1 \leftrightarrow 2\ . \tag{B1e}
$$

The values of the potentials at the location of body 1, following from (3.6), are

$$
\begin{aligned}
(V)_1 = \frac{Gm_2}{r_{12}}\Bigg\{&1 + \frac{1}{c^2}\left[-\frac{3}{2}\frac{Gm_1}{r_{12}} + 2v_2^2 - \frac{1}{2}(n_{12}v_2)^2\right] + \frac{4}{3}\frac{Gm_1}{r_{12}c^3}(n_{12}v_{12})\\
&+\frac{Gm_1}{r_{12}c^4}\Bigg[\frac{11}{2}\frac{Gm_1}{r_{12}} + \frac{5}{4}\frac{Gm_2}{r_{12}} + \frac{15}{8}v_1^2 - \frac{7}{4}(v_1v_2) - \frac{25}{8}v_2^2\\
&\qquad+\frac{1}{8}(n_{12}v_1)^2 - \frac{25}{4}(n_{12}v_1)(n_{12}v_2) + \frac{33}{8}(n_{12}v_2)^2\Bigg]\\
&+\frac{1}{c^4}\left[2v_2^4 - \frac{3}{2}(n_{12}v_2)^2 v_2^2 + \frac{3}{8}(n_{12}v_2)^4\right]\Bigg\}\\
+ \frac{G^2 m_1 m_2}{c^5 r_{12}^2}\Bigg\{&-\frac{2}{5}\frac{Gm_1}{r_{12}}(n_{12}v_1) - \frac{26}{15}\frac{Gm_2}{r_{12}}(n_{12}v_1) + \frac{22}{5}\frac{Gm_1}{r_{12}}(n_{12}v_2)\\
&+\frac{2}{5}\frac{Gm_2}{r_{12}}(n_{12}v_2) - \frac{32}{15}(n_{12}v_1)v_1^2 + \frac{48}{5}(n_{12}v_1)(v_1v_2) - \frac{122}{15}(n_{12}v_1)v_2^2\\
&+\frac{92}{15}(n_{12}v_2)v_1^2 - \frac{184}{15}(n_{12}v_2)(v_1v_2) + \frac{34}{5}(n_{12}v_2)v_2^2 + 2(n_{12}v_1)^3\\
&-10(n_{12}v_1)^2(n_{12}v_2) + 12(n_{12}v_1)(n_{12}v_2)^2 - 4(n_{12}v_2)^3\Bigg\} + O(6)\ ,
\end{aligned} \tag{B2a}
$$

$$
\begin{aligned}
(V_i)_1 = \frac{Gm_2}{r_{12}}\Bigg\{&v_2^i + \frac{v_2^i}{c^2}\left[-2\frac{Gm_1}{r_{12}} + v_2^2 - \frac{1}{2}(n_{12}v_2)^2\right] + \frac{1}{2}\frac{Gm_1}{r_{12}c^2}v_1^i\\
&+\frac{Gm_1}{r_{12}c^2}n_{12}^i\left[-\frac{3}{2}(n_{12}v_1) + \frac{1}{2}(n_{12}v_2)\right]\\
&+\frac{Gm_1}{c^3 r_{12}}v_1^i(n_{12}v_1) - \frac{Gm_1}{c^3 r_{12}}v_2^i\left[\frac{5}{3}(n_{12}v_1) - \frac{2}{3}(n_{12}v_2)\right]\\
&+\frac{Gm_1}{c^3 r_{12}}n_{12}^i\left[\frac{2}{3}\frac{Gm_1}{r_{12}} - \frac{1}{3}\frac{Gm_2}{r_{12}} + v_1^2 - (v_1v_2) - (n_{12}v_{12})^2\right]\Bigg\} + O(4)\ ,
\end{aligned} \tag{B2b}
$$

$$
\begin{aligned}
(\hat{W}_{ij})_1 = \frac{Gm_2}{r_{12}}\Bigg\{&v_2^{ij} - \delta^{ij}v_2^2 + \frac{Gm_1}{r_{12}}\left[-2n_{12}^{ij} + \delta^{ij}\right]\\
&+\frac{Gm_2}{4r_{12}}\left[n_{12}^{ij} - \delta^{ij}\right] + 4\frac{Gm_1}{c r_{12}}n_{12}^{(i}v_{12}^{j)}\\
&-\frac{Gm_1}{c r_{12}}(n_{12}v_{12})\left(3n_{12}^{ij} + \delta^{ij}\right)\Bigg\} + O(2)\ ,
\end{aligned} \tag{B2c}
$$

$$
\begin{aligned}
(\hat{R}_i)_1 = & \frac{G^2 m_1 m_2}{r_{12}^2}\left[-\frac{3}{4}v_1^i + \frac{5}{4}v_2^i - \frac{1}{2}(n_{12}v_1)n_{12}^i - \frac{1}{2}(n_{12}v_2)n_{12}^i\right] \\
& + \frac{G^2 m_2^2}{r_{12}^2}\left[-\frac{1}{8}v_2^i + \frac{1}{8}(n_{12}v_2)n_{12}^i\right] \\
& + \frac{G^2 m_1 m_2}{c r_{12}^2}\Big\{n_{12}^i\left[\frac{1}{4}v_1^2 - \frac{1}{4}v_2^2 - \frac{3}{4}(n_{12}v_1)^2 + \frac{3}{4}(n_{12}v_2)^2\right] \\
& \qquad + \frac{1}{2}(n_{12}v_1)v_1^i - \frac{1}{2}(n_{12}v_2)v_2^i\Big\} + O(2)\ ,
\end{aligned} \tag{B2d}
$$

$$
\begin{aligned}
(\hat{X})_1 = & \frac{G^2 m_1 m_2}{r_{12}^2}\left[-\frac{3}{2}\frac{Gm_1}{r_{12}} + \frac{1}{4}v_1^2 - 2(v_1v_2) + \frac{9}{4}v_2^2\right. \\
& \quad \left. - \frac{11}{4}(n_{12}v_1)^2 + \frac{9}{2}(n_{12}v_1)(n_{12}v_2) - \frac{11}{4}(n_{12}v_2)^2\right] \\
& + \frac{G^2 m_2^2}{r_{12}^2}\left[\frac{1}{12}\frac{Gm_2}{r_{12}} - \frac{1}{8}v_2^2 + \frac{1}{8}(n_{12}v_2)^2\right] \\
& + \frac{G^2 m_1 m_2}{c r_{12}^2}\Big\{-3(n_{12}v_1)^3 + \frac{15}{2}(n_{12}v_1)^2(n_{12}v_2) - \frac{15}{2}(n_{12}v_1)(n_{12}v_2)^2 + 3(n_{12}v_2)^3 \\
& \quad + 3(n_{12}v_1)v_1^2 - \frac{5}{2}(n_{12}v_2)v_1^2 - 5(n_{12}v_{12})(v_1v_2) + \frac{5}{2}(n_{12}v_1)v_2^2 - 3(n_{12}v_2)v_2^2 \\
& \quad - \frac{3}{2}\frac{Gm_1}{r_{12}}(n_{12}v_{12}) - \frac{5}{2}\frac{Gm_2}{r_{12}}(n_{12}v_{12})\Big\} + O(2)\ .
\end{aligned} \tag{B2e}
$$

The gradients of the potentials computed at body 1 (needed for the equations of motion) are

$$
\begin{aligned}
(\partial_i V)_1 = & -\frac{Gm_2}{r_{12}^2}n_{12}^i + \frac{Gm_2}{r_{12}^2c^2}n_{12}^i\left[-2v_2^2 + \frac{3}{2}(n_{12}v_2)^2 + \frac{Gm_1}{r_{12}}\right] - \frac{Gm_2}{r_{12}^2c^2}(n_{12}v_2)v_2^i \\
& + \frac{Gm_2}{r_{12}^2c^4}n_{12}^i\left[-2v_2^4 + \frac{9}{2}v_2^2(n_{12}v_2)^2 - \frac{15}{8}(n_{12}v_2)^4\right] \\
& + \frac{G^2 m_1 m_2}{r_{12}^3c^4}n_{12}^i\left[-\frac{3}{4}v_1^2 - \frac{3}{4}v_2^2 + \frac{7}{2}(v_1v_2) - \frac{13}{2}(n_{12}v_1)^2\right. \\
& \qquad \left. + 7(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 - \frac{1}{4}\frac{Gm_1}{r_{12}} - \frac{1}{2}\frac{Gm_2}{r_{12}}\right] \\
& + \frac{Gm_2}{r_{12}^2c^4}v_2^i\left[-3v_2^2(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^3 - \frac{17}{4}\frac{Gm_1}{r_{12}}(n_{12}v_1) + \frac{9}{4}\frac{Gm_1}{r_{12}}(n_{12}v_2)\right] \\
& + \frac{G^2 m_1 m_2}{r_{12}^3c^4}v_1^i\left[-\frac{9}{4}(n_{12}v_2) + \frac{9}{4}(n_{12}v_1)\right] \\
& + \frac{G^2 m_1 m_2}{c^5 r_{12}^3}n_{12}^i\Big(-10(n_{12}v_1)^3 + 10(n_{12}v_1)^2(n_{12}v_2) + 10(n_{12}v_1)(n_{12}v_2)^2 \\
& \qquad - 10(n_{12}v_2)^3 + \frac{42}{5}(n_{12}v_1)v_1^2 - \frac{22}{5}(n_{12}v_2)v_1^2 - \frac{24}{5}(n_{12}v_1)(v_1v_2) \\
& \qquad - \frac{16}{5}(n_{12}v_2)(v_1v_2) - \frac{18}{5}(n_{12}v_1)v_2^2 + \frac{38}{5}(n_{12}v_2)v_2^2 + \frac{88}{15}\frac{Gm_1}{r_{12}}(n_{12}v_1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{52}{15}\frac{Gm_2}{r_{12}}(n_{12}v_1)-\frac{68}{15}\frac{Gm_1}{r_{12}}(n_{12}v_2)+\frac{24}{5}\frac{Gm_2}{r_{12}}(n_{12}v_2)\Big)\\
&+\frac{G^2m_1m_2}{c^5r_{12}^3}v_1^i\Big(10(n_{12}v_1)^2-8(n_{12}v_1)(n_{12}v_2)-2(n_{12}v_2)^2-\frac{16}{15}\frac{Gm_1}{r_{12}}+\frac{44}{15}\frac{Gm_2}{r_{12}}\\
&\quad -\frac{62}{15}v_1^2+\frac{44}{15}(v_1v_2)+\frac{6}{5}v_2^2\Big)\\
&+\frac{G^2m_1m_2}{c^5r_{12}^3}v_2^i\Big(-6(n_{12}v_1)^2+6(n_{12}v_2)^2+\frac{12}{5}\frac{Gm_1}{r_{12}}-\frac{8}{5}\frac{Gm_2}{r_{12}}+\frac{14}{5}v_1^2\\
&\quad -\frac{4}{15}(v_1v_2)-\frac{38}{15}v_2^2\Big)+O(6)\ ,
\end{aligned}
\tag{B3a}
$$

$$
\begin{aligned}
(\partial_jV_i)_1 &= -\frac{Gm_2}{r_{12}^2}n_{12}^jv_2^i+\frac{Gm_2}{r_{12}^2c^2}\Big\{n_{12}^{ij}\frac{Gm_1}{r_{12}}\Big[-\frac{3}{2}(n_{12}v_1)+\frac{5}{2}(n_{12}v_2)\Big]\\
&\quad +\frac{Gm_1}{2r_{12}}n_{12}^jv_1^i+n_{12}^jv_2^i\Big[-v_2^2+\frac{3}{2}(n_{12}v_2)^2+\frac{Gm_1}{2r_{12}}\Big]\\
&\quad -\frac{Gm_1}{r_{12}}n_{12}^iv_2^j-v_2^{ij}(n_{12}v_2)\Big\}\\
&+\frac{G^2m_1m_2}{c^3r_{12}^3}\Big\{n_{12}^{ij}\Big(-5(n_{12}v_{12})^2+v_{12}^2+\frac{1}{3}\frac{Gm_1}{r_{12}}+\frac{1}{3}\frac{Gm_2}{r_{12}}\Big)\\
&\quad +6n_{12}^{(i}v_1^{j)}(n_{12}v_{12})-6n_{12}^{(i}v_2^{j)}(n_{12}v_{12})-\frac{4}{3}v_{12}^{ij}\Big\}+O(4)\ ,
\end{aligned}
\tag{B3b}
$$

$$
\begin{aligned}
(\partial_k\hat{W}_{ij})_1 &= \frac{Gm_2}{r_{12}^2}\Big\{-n_{12}^kv_2^{ij}+\delta^{ij}n_{12}^k\Big[-\frac{Gm_1}{2r_{12}}+\frac{Gm_2}{2r_{12}}+v_2^2\Big]\\
&\quad +\delta^{k(i}n_{12}^{j)}\Big[-\frac{3Gm_1}{2r_{12}}+\frac{Gm_2}{2r_{12}}\Big]+n_{12}^{ijk}\Big[\frac{2Gm_1}{r_{12}}-\frac{Gm_2}{r_{12}}\Big]\Big\}+O(2)\ ,
\end{aligned}
\tag{B3c}
$$

$$
\begin{aligned}
(\partial_j\hat{R}_i)_1 &= \frac{Gm_2}{r_{12}^2}\Big\{\delta^{ij}\Big[-\frac{5}{8}\frac{Gm_1}{r_{12}}(n_{12}v_1)+\frac{1}{4}\frac{Gm_1}{r_{12}}(n_{12}v_2)+\frac{1}{8}\frac{Gm_2}{r_{12}}(n_{12}v_2)\Big]\\
&\quad +n_{12}^{ij}\Big[\frac{1}{2}\frac{Gm_1}{r_{12}}(n_{12}v_1)+\frac{1}{2}\frac{Gm_1}{r_{12}}(n_{12}v_2)-\frac{1}{2}\frac{Gm_2}{r_{12}}(n_{12}v_2)\Big]\\
&\quad +n_{12}^i\Big[\frac{1}{4}\frac{Gm_1}{r_{12}}v_1^j-\frac{5}{8}\frac{Gm_1}{r_{12}}v_2^j+\frac{1}{8}\frac{Gm_2}{r_{12}}v_2^j\Big]\\
&\quad +n_{12}^j\Big[\frac{7}{8}\frac{Gm_1}{r_{12}}v_1^i-\frac{9}{8}\frac{Gm_1}{r_{12}}v_2^i+\frac{1}{4}\frac{Gm_2}{r_{12}}v_2^i\Big]\Big\}+O(2)\ ,
\end{aligned}
\tag{B3d}
$$

$$
\begin{aligned}
(\partial_i\hat{X})_1 &= \frac{G^2m_1m_2}{r_{12}^3}\Big\{n_{12}^i\Big[\frac{1}{2}\frac{Gm_1}{r_{12}}-\frac{3}{2}\frac{Gm_2}{r_{12}}+\frac{9}{2}(n_{12}v_1)^2-8(n_{12}v_1)(n_{12}v_2)\\
&\qquad +\frac{9}{2}(n_{12}v_2)^2+\frac{7}{4}(v_1v_2)-2v_2^2\Big]\\
&\quad -\frac{5}{2}(n_{12}v_1)v_1^i+3(n_{12}v_2)v_1^i+\frac{5}{4}(n_{12}v_1)v_2^i-\frac{5}{2}(n_{12}v_2)v_2^i\Big\}\\
&+\frac{G^2m_2^2}{r_{12}^3}\Big\{n_{12}^i\Big[-\frac{1}{4}\frac{Gm_2}{r_{12}}-\frac{1}{2}(n_{12}v_2)^2+\frac{1}{4}v_2^2\Big]+\frac{1}{4}(n_{12}v_2)v_2^i\Big\}\\
&+\frac{G^3m_1m_2^2}{cr_{12}^4}\Big\{3n_{12}^i(n_{12}v_{12})-2v_{12}^i\Big\}+O(2)\ .
\end{aligned}
\tag{B3e}
$$

REFERENCES

[1] H. Shinkai, W.-M. Suen, F.D. Swesty, M. Tobias, E.Y.M. Wang and C.M. Will, in *the Proc. of the 8th Marcel Grossmann Meeting*, Jerusalem 1997, to appear.

[2] P.R. Brady, J.D.E. Creighton and K.S. Thorne, "Computing the merger of black-hole binaries : the IBBH problem", gr-qc/9804057.

[3] L. Bel, T. Damour, N. Deruelle, J. Ibañez and J. Martin, Gen. Relativ. Gravit. **13**, 963 (1981).

[4] T. Damour and N. Deruelle, Phys. Lett. **87A**, 81 (1981).

[5] N. Deruelle, thèse de doctorat d'état, Paris (1982).

[6] T. Damour, C. R. Acad. Sc. Paris, **294**, 1355 (1982).

[7] T. Damour, in *Gravitational Radiation*, N. Deruelle and T. Piran (eds.), (North-Holland publishing Company, 1983), p. 59.

[8] P. Jaranowski and G. Schäfer, Phys. Rev. D**57**, 7274 (1998).

[9] J.H. Taylor, L.A. Fowler and P.M. Mc Culloch, Nature **277**, 437 (1979).

[10] J.H. Taylor and J.M. Weisberg, Astrophys. J. **253**, 908 (1982).

[11] T. Damour and J.H. Taylor, Astrophys. J. **366**, 501 (1991).

[12] J.H. Taylor, Class. Quantum Grav. **10**, S167 (1993).

[13] C. Cutler, T.A. Apostolatos, L. Bildsten, L.S. Finn, E.E. Flanagan, D. Kennefick, D.M. Markovic, A. Ori, E. Poisson, G.J. Sussman and K.S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).

[14] C. Cutler, L.S. Finn, E. Poisson and G.J. Sussman, Phys. Rev. D**47**, 1511 (1993).

[15] L.S. Finn and D.F. Chernoff, Phys. Rev. D**47**, 2198 (1993).

[16] C. Cutler and E. Flanagan, Phys. Rev. D**49**, 2658 (1994).

[17] H. Tagoshi and T. Nakamura, Phys. Rev. D**49**, 4016 (1994).

[18] E. Poisson, Phys. Rev. D**52**, 5719 (1995); D**55**, 7980 (1997).

[19] K. Westpfahl and M. Göller, Lett Nuov. Cim. **26**,573 (1979).

[20] J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris: Hermann (1932).

[21] A. Sellier, Proc. R. Soc. Lond. A**445**, 69 (1994).

[22] A.S. Eddington, *The mathematical theory of relativity* (1923), reprinted by Cambridge University Press (1965).

[23] M. Walker and C.M. Will, Astrophys. J. **242**, L129 (1980).

[24] M. Riesz, Acta Mathematica, **81**, 1 (1949).

[25] Some poles are expected to develop at the next 3PN approximation, corresponding to the appearance of logarithmic terms in the equations of motion.

[26] T. Damour and N. Deruelle, C. R. Acad. Sc. Paris **293**, 537 (1981).

[27] T. Damour and G. Schäfer, Gen. Rel. Grav. **17**, 879 (1985).

[28] T. Damour, Phys. Rev. Lett. **51**, 1019 (1983).

[29] T. Ohta, H. Okamura, T. Kimura and K. Hiida, Progr. Theor. Phys. **51**, 1220 (1974).

[30] T. Ohta, H. Okamura, T. Kimura and K. Hiida, Progr. Theor. Phys. **51**, 1598 (1974).

[31] T. Ohta, H. Okamura, T. Kimura and K. Hiida, Progr. Theor. Phys. **50**, 492 (1973).

[32] T. Ohta, T. Kimura and K. Hiida, Nuov. Cim. B**27**, 103 (1975).

[33] T. Damour, L.P. Grishchuk, S.M. Kopejkin and G. Schäfer, in the Proc. of the *5th Marcel Grossmann meeting on General Relativity*, R. Ruffini (ed.), p. 451 (1988).

[34] G. Schäfer, Ann. Phys. (N.Y.) **161**, 81 (1985).

[35] G. Schäfer, Gen. Rel. Grav. **18**, 255 (1986).

[36] P. Jaranowski and G. Schäfer, Phys. Rev. D**55**, 4712 (1997).

[37] S.M. Kopejkin, Astron. Zh. **62**, 889 (1985).

[38] L.P. Grishchuk and S.M. Kopejkin, in *Relativity in Celestial Mechanics and Astrometry*, J. Kovalevsky and V.A. Brumberg (eds.), Reidel, Dordrecht (1986).

[39] L. Blanchet, Phys. Rev. D**51**, 2559 (1995).

[40] The post-Newtonian metric though valid formally everywhere in space-time is expected to constitute a good approximation to an exact solution only in the near zone of the source, i.e. in a region of much smaller scale than the wavelength of the emitted radiation. For inspiralling compact binaries, the near-zone covers entirely the binary's orbit.

[41] T. Damour, unpublished calculations (1981).

[42] L. Blanchet, T. Damour and B.R. Iyer, Phys. Rev. D**51**, 5360 (1995).

[43] Greek indices take the values 0,1,2,3, and Latin 1,2,3. Our signature is (−+++).

[44] L. Blanchet and T. Damour, Ann. Inst. H. Poincaré (Phys. Théorique) **50**, 377 (1989).

[45] Indeed, using the equation of continuity (2.2a), we see that there is no term $O(1)$.

[46] We have $\hat{W}_{ij} = W_{ij} - \delta_{ij} W_{kk}$, where W_{ij} is the definition adopted in [39].

[47] For the present application to 2.5PN order, the coefficient of the Dirac function $\delta(\mathbf{x}-\mathbf{y}_1)$ in the standard expression of the stress-energy tensor $T^{\mu\nu}$ can be replaced by its value at $\mathbf{x} = \mathbf{y}_1$.

[48] P. Jaranowski, in *Mathematics of Gravitation*, A. Królak (ed.) (Banach Center Publications, 1997).

[49] Since all infinite terms are removed the Hadamard regularization does not need to be followed by a renormalization.

[50] A less trivial example (not needed in this paper) is $(V^4)_1 = G^4 m_2^2(2m_1^2+m_2^2)/r_{12}^4+O(2)$, showing that the Hadamard partie finie is in general not "distributive" with respect to multiplication, i.e. $(V^4)_1$ is not equal to $[(V)_1]^4$.

[51] R. Estrada and R. P. Kanwal, Proc. R. Soc. Lond. A**401**, 294 (1985).

[52] Note that this solution represents only a particular solution of the equation we want to solve. However, any possible homogeneous solution must be regular in $\mathbf{x}$ and in the individual source points $\mathbf{y}_{1,2}$, and must have a compatible dimension. One can check that the only possibility is to add to g a simple numerical constant. This constant disappears after application of the spatial derivatives present in front of the term.

[53] Note that the Hadamard partie finie of the divergent integral is related to the value of $r^2\partial_r g$ at infinity [say $(r^2\partial_r g)_\infty$], as computed in the same way as for the Hadamard partie finie of a function at some finite-distance point, i.e. like in (3.6) by expanding $r^2\partial_r g$ when $r \to \infty$ and taking the average over $\mathbf{n}$ of the term with zeroth power of r. We get: Partie finie $\left\{-\frac{1}{4\pi}\int d^3\mathbf{x}/r_1 r_2\right\} = -(r^2\partial_r g)_\infty = r_{12}/2$. There is agreement with the value of the function Y defined by analytic continuation in the equations (4.22) and (4.23) of [39] for $\ell = 0$.

[54] M. Carmeli, Nuov. Cim. **37**, 842 (1965).

[55] G. Schäfer, Phys. Lett. A**123**, 336 (1987).

[56] The solutions K_1 and H_1 satisfy the Poisson equations (6.3) in the sense of distributions, and tend to zero at infinity ($|\mathbf{x}| \to +\infty$).

[57] The Hadamard partie finie of the divergent integral $\int d^3\mathbf{x}/r_1^5$ is zero.

[58] Note that many of the "odd" terms in the metric, which are associated with radiation-

reaction, are only functions of time t or depend smoothly on the field point $\mathbf{x}$ (through for instance the product $r_1\mathbf{n}_1 = \mathbf{x} - \mathbf{y}_1$). However, when presented into a form which shows this fact more explicitly (expressing all the terms with the help of the vectors $\mathbf{r}_1$ and $\mathbf{r}_{12}$) the metric is not simpler.

[59] We have also $(nv_{12}) = r_{12}\ \omega_{2\mathrm{PN}} \sin i \cos \phi_{2\mathrm{PN}} + O(5)$, where $\phi_{2\mathrm{PN}}$ denotes the orbital phase and i the inclination angle.